

Information Geometry of Quantum Entangled Gaussian Wave-Packets

D.-H. Kim*

*Institute for the Early Universe, Ewha Womans University, Seoul 120-750, South Korea and
International Institute for Theoretical Physics and Mathematics Einstein-Galilei, via Santa Gonda 14, 59100 Prato, Italy*

S. A. Ali

*International Institute for Theoretical Physics and Mathematics Einstein-Galilei, via Santa Gonda 14, 59100 Prato, Italy
Department of Physics, State University of New York at Albany,
1400 Washington Avenue, Albany, NY 12222, USA and
Department of Arts and Sciences, Albany College of Pharmacy and Health Sciences,
106 New Scotland Avenue, Albany, NY 12208, USA*

C. Cafaro and S. Mancini

*School of Science and Technology, Physics Division,
University of Camerino, I-62032 Camerino, Italy*

Describing and understanding the essence of quantum entanglement and its connection to dynamical chaos is of great scientific interest. In this work, using information geometric (IG) techniques, we investigate the effects of micro-correlations on the evolution of maximal probability paths on statistical manifolds induced by systems whose microscopic degrees of freedom are Gaussian distributed. We use the statistical manifolds associated with correlated and non-correlated Gaussians to model the scattering induced quantum entanglement of two spinless, structureless, non-relativistic particles, the latter represented by minimum uncertainty Gaussian wave-packets. Knowing that the degree of entanglement is quantified by the purity \mathcal{P} of the system, we express the purity for s -wave scattering in terms of the micro-correlation coefficient r - a quantity that parameterizes the correlated microscopic degrees of freedom of the system; thus establishing a connection between entanglement and micro-correlations. Moreover, the correlation coefficient r is readily expressed in terms of physical quantities involved in the scattering, the precise form of which is obtained via our IG approach. It is found that the entanglement duration can be controlled by the initial momentum p_0 , momentum spread σ_0 and r . Furthermore, we obtain exact expressions for the IG analogue of standard indicators of chaos such as the sectional curvatures, Jacobi field intensities and the Lyapunov exponents. We then present an analytical estimate of the information geometric entropy (IGE); a suitable measure that quantifies the complexity of geodesic paths on curved manifolds. Finally, we present concluding remarks addressing the usefulness of an IG characterization of both entanglement and complexity in quantum physics.

PACS numbers: Probability Theory (02.50.Cw), Riemannian Geometry (02.40.Ky), Complexity (89.70.Eg), Entropy (89.70.Cf), Quantum Entanglement (03.65.Ud).

I. INTRODUCTION

One of the most debated features of composite quantum mechanical systems is their ability to become entangled [1, 2]. By quantum entanglement we mean quantum correlations among the distinct subsystems of the entire composite quantum system. For such correlated quantum systems, it is not possible to specify the quantum state of any subsystem independently of the remaining subsystems.

The generation of quantum entanglement among spatially separated particles requires non-local interactions through which quantum correlations are dynamically created [3–5].

Quantum entanglement is an indispensable resource for quantum information processes [6]. Continuous Variable Quantum Systems (CVQS) are also an interesting topic in quantum information theory [7]. By CVQS we refer to quantum mechanical systems on which one can - in principle - perform measurements of certain observables whose eigenvalue spectrum is continuous. Examples of CVQS are the quantized motion of massive particles with the corresponding position and momentum observables and the quantized mode of the electromagnetic field with its quadrature observables among others. Most examples of possible applications of entanglement in continuous variable

*Electronic address: ki1313@yahoo.com

quantum information is based on EPR states [1] or in the optical case, approximations of EPR states using squeezed states [8–10]. Continuous variable entanglement has been investigated in the context of photon-atom scattering [11], photoionization processes [12–15], trapped atoms [16], and classically chaotic systems [17, 18]. Correlated CVQS can be used as an invaluable non-classical resource for quantum computation and quantum communication [7].

The most realistic approach to the generation of entangled continuous variable systems is via dynamical interaction, of which local scattering events (collisions) are a natural, ubiquitous type [19]. Scattering can result in a decomposition of the wave function into transmission and reflection modes. Due to the mutual interactions present in scattering processes (such as interference between incoming and reflected parts of the wave function of the composite system), quantum particles can become entangled. Moreover, scattering may result in a distortion (due to rapid fluctuation of scattering amplitude with relative momentum for instance [3, 20]) of the shape of the wave function. In cases with constant amplitudes the wave function of the system may be rendered inseparable as a consequence of reflection induced distortion. Interference between incoming and reflected parts of the wave function of the system or the distortion effect can result in a non-separable post collision two-particle state. Entanglement generation in non-relativistic scattering of distinguishable particles has been investigated by a number of researchers [3, 19–26]. Most treatments consider interactions among similar particle types [3, 19–21, 24, 25]. It is however, unclear as to how the interaction (scattering) potentials and incident particle energies control the strength of entanglement [3]. As will be seen in what follows, the information geometric approach employed in the present work lends some degree of clarification on this issue. Furthermore, describing and understanding the complexity of quantum processes is still an open problem and our present knowledge on the relations among complexity, chaoticity and quantum entanglement are not at all satisfactory [27]. As we will see, our work sheds some lights on this issue as well.

In this article, we explore the potential utility of the Information Geometric Approach to Chaos (IGAC) [28–33] for analyzing quantum mechanical systems. The IGAC is a theoretical framework developed to study chaos in informational geodesic flows on statistical manifolds associated with probabilistic descriptions of physical systems.

We seek to provide a quantitative estimate of the degree of entanglement of CVQS in terms of information geometric quantities such as solutions to geodesic equations (expected values of momentum and momentum spread in this case) and micro-correlation coefficient r . The quantity r parameterizes the correlated microscopic degrees of freedom of the system. An important question that arises is whether or not r can be understood in terms of physically measurable quantities.

As described above, when particles with no initial correlations collide, they may emerge from the interaction entangled [34]. Hence, we consider two CVQS with Gaussian continuous degrees of freedom that are prepared independently, interact via a scattering process mediated by an interaction (scattering) potential with finite range and separate again. We investigate the entanglement of the two-particle wave function of the system generated by such a scattering event. In this context we ask the question: how much entanglement between the two particles is generated and on what does it depend? Surprisingly, there are only a few studies of entanglement production from the scattering of two particles [35, 36]. The nature of quantum entanglement arising from s -wave scattering has yet to be fully explored [37]. We choose to use Gaussian states [36, 38] since many important properties of these states can often be obtained in an analytic fashion. Moreover, it is known that a good way to describe naturally occurring quantum states is as spatially localized Gaussian wave-packets, or as density matrices built from them [23, 39–42].

For a system of two spinless, structureless, non-relativistic particles with no internal degrees of freedom, a complete set of commuting observables is furnished by the momentum operators of each particle [34]. The continuous variables in our case are therefore taken to be the momentum of each particle. We investigate how the initial conditions and the magnitude of r of the system affect entanglement; specifically, the duration of entanglement. By duration of entanglement we mean the temporal behavior of the magnitude of entanglement. The IGAC is also used to estimate the extent to which the complexity of the geodesic information flow (continuous spectrum of expected values of some relevant observable) of the quantum system is affected by r .

The layout of this article is as follows. In Section II, we reexamine the s -wave scattering induced quantum entanglement of two spinless, structureless, non-relativistic particles, which are represented by two-particle Gaussian wave-packets [24]. We exploit the fact that the three-dimensional scattering problem can be effectively reduced to a one-dimensional problem as far as the three-dimensional representations of wave-packets are isotropic. In Section III, we outline the main ideas behind the IGAC and present the information geometry of correlated and uncorrelated Gaussian statistical manifolds, which is to be employed in our investigation of quantum entanglement and complexity in the following sections. In Section IV, we use information geometric techniques in conjunction with standard partial wave quantum scattering theory to provide an information geometric characterization of quantum entanglement. In Section V, we obtain exact expressions for the information geometric analogue of standard indicators of chaos such as sectional curvatures, Jacobi field intensities and Lyapunov exponents. Finally, we present an analytical estimate of the information geometric entropy (IGE) and this allows us to connect quantum entanglement to the complexity of informational geodesic flows in a quantitative manner. Our concluding remarks are presented in Section VI.

II. ENTANGLEMENT AND GAUSSIAN WAVE-PACKET SCATTERING PROCESSES

In this Section, we reexamine the s -wave scattering induced quantum entanglement of two spinless, structureless, non-relativistic particles, represented by two-particle Gaussian wave-packets as presented in [24]. For ease of analysis, we exploit the fact that the three-dimensional scattering problem can be effectively reduced to a one-dimensional problem as the three-dimensional representations of wave-packets are isotropic.

A. The Pre-collisional Scenario and the Effective Dimensional Reduction

For the purpose of modeling a head-on collision we consider two identical (but distinguishable), spinless particles in momentum space, each represented by minimum uncertainty Gaussian wave-packets. Before collision, particles 1 and 2 are initially located far from each other - a linear distance \mathbf{R}_o - each having the initial average momentum $\langle \mathbf{p}_1 \rangle_o = \mathbf{p}_o$ and $\langle \mathbf{p}_2 \rangle_o = -\mathbf{p}_o$, respectively, with equal momentum dispersion σ_o (see Figure 1). The normalized, separable (i.e., non-entangled) two-particle Gaussian wave function representing the situation before collision is then given by [24]

$$\psi(\mathbf{k}_1, \mathbf{k}_2) = \psi_1(\mathbf{k}_1) \otimes \psi_2(\mathbf{k}_2), \quad (1)$$

with respective single particle wave functions

$$\psi_{1/2}(\mathbf{k}_{1/2}) = a(\mathbf{k}_{1/2}, \langle \mathbf{k}_{1/2} \rangle_o; \sigma_{ko}) e^{i(\mathbf{k}_{1/2} - \langle \mathbf{k}_{1/2} \rangle_o) \cdot \mathbf{q}_{1/2}}, \quad (2)$$

where

$$a(\mathbf{k}_{1/2}, \langle \mathbf{k}_{1/2} \rangle_o; \sigma_{ko}) \equiv \left(\frac{1}{2\pi\sigma_{ko}^2} \right)^{3/4} \exp \left[-\frac{(\mathbf{k}_{1/2} - \langle \mathbf{k}_{1/2} \rangle_o)^2}{4\sigma_{ko}^2} \right], \quad (3)$$

with $\mathbf{k}_{1/2} = \frac{\mathbf{p}_{1/2}}{\hbar}$, $\langle \mathbf{k}_{1/2} \rangle_o = \frac{\langle \mathbf{p}_{1/2} \rangle_o}{\hbar} = \pm \frac{\mathbf{p}_o}{\hbar} = \pm \mathbf{k}_o$, $\sigma_{ko} = \frac{\sigma_o}{\hbar}$ and $\mathbf{q}_{1/2} = \mp \frac{1}{2} \mathbf{R}_o$. Observe that wave functions in (2) satisfy the normalization conditions

$$\int \psi_1^*(\mathbf{k}_1) \psi_1(\mathbf{k}_1) d^3\mathbf{k}_1 = \int a^2(\mathbf{k}_1, \langle \mathbf{k}_1 \rangle_o; \sigma_{ko}) d^3\mathbf{k}_1 = 1 \quad (4)$$

and

$$\int \psi_2^*(\mathbf{k}_2) \psi_2(\mathbf{k}_2) d^3\mathbf{k}_2 = \int a^2(\mathbf{k}_2, \langle \mathbf{k}_2 \rangle_o; \sigma_{ko}) d^3\mathbf{k}_2 = 1. \quad (5)$$

The type of state described by (1) is ubiquitous when describing quantum systems of continuous variables.

One should note that the three-dimensional Gaussian wave-packet (1) is isotropic. That is to say, in polar coordinates the representation of the separable two-particle state exhibits a functional dependence on the radial variable only. For this reason the three-dimensional vectorial representation (1) may be effectively reduced to a one-dimensional representation. This may be demonstrated as follows. First, using (1), (2) and (3), we express the two-particle wave-packet in Cartesian coordinates as

$$\psi((k_1^x, k_1^y, k_1^z), (k_2^x, k_2^y, k_2^z)) = \psi_1(k_1^x, k_1^y, k_1^z) \otimes \psi_2(k_2^x, k_2^y, k_2^z), \quad (6)$$

where

$$\psi_{1/2}(k_{1/2}^x, k_{1/2}^y, k_{1/2}^z) = \left(\frac{1}{2\pi\sigma_{ko}^2} \right)^{3/4} \prod_{j=x,y,z} \exp \left[-\frac{(k_{1/2}^j - \langle k_{1/2}^j \rangle_o)^2}{4\sigma_{ko}^2} \right] e^{i(k_{1/2}^j - \langle k_{1/2}^j \rangle_o) q_{1/2}^j}, \quad (7)$$

with $k_{1/2}^j = \mathbf{k}_{1/2} \cdot \mathbf{e}_j$, $\langle k_{1/2}^j \rangle_o = \langle \mathbf{k}_{1/2} \rangle_o \cdot \mathbf{e}_j$, $q_{1/2}^j = \mathbf{q}_{1/2} \cdot \mathbf{e}_j$, the components in a Cartesian basis $\{\mathbf{e}_j\}$. Then the probability density for this wave-packet becomes

$$\begin{aligned} |\psi(\mathbf{k}_1, \mathbf{k}_2)|^2 &= \psi^*((k_1^x, k_1^y, k_1^z), (k_2^x, k_2^y, k_2^z)) \psi((k_1^x, k_1^y, k_1^z), (k_2^x, k_2^y, k_2^z)) \\ &= \left(\frac{1}{2\pi\sigma_{ko}^2} \right)^3 \prod_{j=x,y,z} \exp \left[-\frac{(k_1^j - \langle k_1^j \rangle_o)^2}{2\sigma_{ko}^2} \right] \exp \left[-\frac{(k_2^j - \langle k_2^j \rangle_o)^2}{2\sigma_{ko}^2} \right]. \end{aligned} \quad (8)$$

Figure 1: An illustration of the two-particle system before and after a head-on collision. Before collision the two particles are initially the distance \mathbf{R}_o away from each other and move toward each other with the momenta \mathbf{p}_o and $-\mathbf{p}_o$, respectively; both particles have the identical momentum spread σ_o (\hbar/σ_o in configuration space). After collision a large spherical shell represents the scattered part of the single particle density (i.e. either particle 1 or particle 2) under the s -wave approximation. The arrows indicate that after collision the two particles move away from each other with the momenta, $-\mathbf{p}_o$ and \mathbf{p}_o , respectively [24].

Upon integrating $|\psi(\mathbf{k}_1, \mathbf{k}_2)|^2$ over $d^3\mathbf{k}_1 d^3\mathbf{k}_2$, one obtains

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \prod_{j=x,y,z} dk_1^j dk_2^j |\psi(\mathbf{k}_1, \mathbf{k}_2)|^2 = I_1 I_2, \quad (9)$$

where

$$I_l \equiv \left(\frac{1}{2\pi\sigma_{ko}^2} \right)^{3/2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \prod_{j=x,y,z} dk_l^j \exp \left[-\frac{(k_l^j - \langle k_l^j \rangle_o)^2}{2\sigma_{ko}^2} \right] \text{ with } l = 1, 2. \quad (10)$$

Converting the integrals I_l ($l = 1, 2$) into polar-coordinate representation yields

$$\begin{aligned} I_l &= \left(\frac{1}{2\pi\sigma_{ko}^2} \right)^{3/2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \prod_{j=x,y,z} dk_l^j \exp \left[-\frac{(k_l^j)^2}{2\sigma_{ko}^2} \right] \\ &= \left(\frac{1}{2\pi\sigma_{ko}^2} \right)^{3/2} 4\pi \int_0^{+\infty} dk_l k_l^2 \exp \left(-\frac{k_l^2}{2\sigma_{ko}^2} \right), \end{aligned} \quad (11)$$

where $k_l^2 \equiv \mathbf{k}_l^2 = (k_l^x)^2 + (k_l^y)^2 + (k_l^z)^2$ ($l = 1, 2$). The remaining integral in (11) can be replaced with

$$\begin{aligned} \int_0^{+\infty} dk_l k_l^2 \exp \left(-\frac{k_l^2}{2\sigma_{ko}^2} \right) &= \sigma_{ko}^2 \int_0^{+\infty} dk_l \exp \left(-\frac{k_l^2}{2\sigma_{ko}^2} \right) \\ &= \frac{\sigma_{ko}^2}{2} \int_{-\infty}^{+\infty} dk_l \exp \left(-\frac{k_l^2}{2\sigma_{ko}^2} \right) = \frac{\sigma_{ko}^2}{2} \int_{-\infty}^{+\infty} dk_l \exp \left[-\frac{(k_l - \langle k_l \rangle_o)^2}{2\sigma_{ko}^2} \right], \end{aligned} \quad (12)$$

where we have changed the domain of k_l from $[0, +\infty)$ to $(-\infty, +\infty)$ in order to obtain the last equality. By substituting (12) into (11), followed by substituting (11) into the right-hand side of (9) and finally inserting (8) into

the left-hand side of (9), we establish

$$\begin{aligned} & \left(\frac{1}{2\pi\sigma_{ko}^2} \right)^3 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \prod_{l=1,2} \prod_{j=x,y,z} dk_l^j \exp \left[-\frac{(k_l^j - \langle k_l^j \rangle_o)^2}{2\sigma_{ko}^2} \right] \\ &= \frac{1}{2\pi\sigma_{ko}^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \prod_{l=1,2} dk_l \exp \left[-\frac{(k_l - \langle k_l \rangle_o)^2}{2\sigma_{ko}^2} \right]. \end{aligned} \quad (13)$$

The integration of $|\psi(\mathbf{k}_1, \mathbf{k}_2)|^2$ over $d^3\mathbf{k}_1 d^3\mathbf{k}_2$ now reads

$$\begin{aligned} \iint |\psi(\mathbf{k}_1, \mathbf{k}_2)|^2 d^3\mathbf{k}_1 d^3\mathbf{k}_2 &= \left(\frac{1}{2\pi\sigma_{ko}^2} \right)^3 \iint \exp \left[-\frac{(\mathbf{k}_1 - \langle \mathbf{k}_1 \rangle_o)^2 + (\mathbf{k}_2 - \langle \mathbf{k}_2 \rangle_o)^2}{2\sigma_{ko}^2} \right] d^3\mathbf{k}_1 d^3\mathbf{k}_2 \\ &= \frac{1}{2\pi\sigma_{ko}^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp \left[-\frac{(k_1 - \langle k_1 \rangle_o)^2 + (k_2 - \langle k_2 \rangle_o)^2}{2\sigma_{ko}^2} \right] dk_1 dk_2. \end{aligned} \quad (14)$$

Thus, we may effectively reduce the three dimensional two-particle wave function $\psi(\mathbf{k}_1, \mathbf{k}_2)$ expressed via (1), (2) and (3) to the two-particle one-dimensional wave function $\psi(k_1, k_2)$ given by

$$\psi(k_1, k_2) = \psi_1(k_1) \otimes \psi_2(k_2), \quad (15)$$

with respective single-particle wave functions,

$$\psi_{1/2}(k_{1/2}) = a(k_{1/2}, \langle k_{1/2} \rangle_o; \sigma_{ko}) e^{i(k_{1/2} - \langle k_{1/2} \rangle_o)q_{1/2}}, \quad (16)$$

where

$$a(k_{1/2}, \langle k_{1/2} \rangle_o; \sigma_{ko}) \equiv \left(\frac{1}{2\pi\sigma_{ko}^2} \right)^{1/4} \exp \left[-\frac{(k_{1/2} - \langle k_{1/2} \rangle_o)^2}{4\sigma_{ko}^2} \right], \quad (17)$$

and $k_{1/2} = \frac{p_{1/2}}{\hbar} \in (-\infty, +\infty)$, $\langle k_{1/2} \rangle_o = \frac{\langle p_{1/2} \rangle_o}{\hbar} = \pm \frac{p_o}{\hbar} = \pm k_o$, $\sigma_{ko} = \frac{\sigma_o}{\hbar}$, $q_{1/2} = \mp \frac{1}{2}R_o$. The wave functions (16) satisfy the normalization conditions

$$\int_{-\infty}^{+\infty} \psi_{1/2}^*(k_{1/2}) \psi_{1/2}(k_{1/2}) dk_{1/2} = \int_{-\infty}^{+\infty} a^2(k_{1/2}, \langle k_{1/2} \rangle_o; \sigma_{ko}) dk_{1/2} = 1. \quad (18)$$

B. The Post-collisional Scenario

After collision, the wave function for the two-particle system in the long time limit takes the form [24]:

$$\psi(\mathbf{k}_1, \mathbf{k}_2, t) = (N)^{-1/2} \left[\psi_1(\mathbf{k}_1) \psi_2(\mathbf{k}_2) e^{-i\hbar(k_1^2 + k_2^2)t/(2m)} + \varepsilon \psi_{\text{scat}}(\mathbf{k}_1, \mathbf{k}_2, t) \right], \quad (19)$$

where N and ε are normalization constants such that ψ and ψ_{scat} are both normalized to unity, m denotes the mass of each particle. Here $\psi_{1/2}(\mathbf{k}_{1/2})$ is given by (2) and (3) and thus we have

$$\begin{aligned} \psi_1(\mathbf{k}_1) \psi_2(\mathbf{k}_2) e^{-i\hbar(k_1^2 + k_2^2)t/(2m)} &= \left(\frac{1}{2\pi\sigma_{ko}^2} \right)^{3/2} \exp \left[-\frac{(\mathbf{k}_1 - \mathbf{k}_o)^2 + (\mathbf{k}_2 + \mathbf{k}_o)^2}{4\sigma_{ko}^2} \right] \\ &\times e^{-i(k_1 - k_o)R_o/2 + i(k_2 + k_o)R_o/2 - i\hbar(k_1^2 + k_2^2)t/(2m)}. \end{aligned} \quad (20)$$

We treat $|\varepsilon| \ll 1$ as a small number. Following [24], one can write

$$\varepsilon \psi_{\text{scat}}(\mathbf{k}_1, \mathbf{k}_2, t) = \varepsilon \psi_{\text{c.m.}}(\mathbf{K}, t) \eta(\mathbf{k}, t), \quad (21)$$

where

$$\psi_{\text{c.m.}}(\mathbf{K}, t) = \left(\frac{1}{4\pi\sigma_{k_0}^2} \right)^{3/4} \exp \left(-\frac{\mathbf{K}^2}{8\sigma_{k_0}^2} \right) e^{-i\hbar K^2 t/(2M)}, \quad (22)$$

and the scattering part $\varepsilon\eta(\mathbf{k}, t)$ is given by

$$\varepsilon\eta(\mathbf{k}, t) = \frac{1}{(2\pi)^3} \iint \psi_{\text{rel}}(\mathbf{k}', 0) f(k') \frac{e^{ik'r}}{r} e^{-i\mathbf{k}\cdot\mathbf{r} - i\hbar k'^2 t/(2\mu)} d^3\mathbf{k}' d^3\mathbf{r} \quad (23)$$

with

$$\psi_{\text{rel}}(\mathbf{k}', 0) = \left(\frac{1}{\pi\sigma_{k_0}^2} \right)^{3/4} \exp \left[-\frac{(\mathbf{k}' - k_0\hat{\mathbf{k}})^2}{2\sigma_{k_0}^2} \right] e^{-i(k-k_0)R_0}. \quad (24)$$

Here we have adopted the center of mass and relative coordinates such that the conjugate momenta $\mathbf{K} \equiv \mathbf{k}_1 + \mathbf{k}_2$ and $\mathbf{k} \equiv \frac{1}{2}(\mathbf{k}_1 - \mathbf{k}_2)$ are used along with the total mass, $M = 2m$ and the reduced mass $\mu = m/2$. The quantity $f(k) \equiv \frac{e^{i2\theta(k)} - 1}{2ik}$ is the s -wave scattering amplitude due to the s -wave scattering phase shift $\theta(k)$. Inserting (24) into (23) and performing the integral, we obtain

$$\varepsilon\eta(\mathbf{k}, t) \approx \left(\frac{1}{\pi\sigma_{k_0}^2} \right)^{3/4} \exp \left[-\frac{(\mathbf{k} - k_0\hat{\mathbf{k}})^2}{2\sigma_{k_0}^2} \right] \varrho(k) e^{-i(k-k_0)R_0 - i\hbar k^2 t/(2\mu)}, \quad (25)$$

where

$$\varrho(k) \equiv \frac{4i(k_0 - i\sigma_{k_0}^2 R_0) k^2 f(k)}{\sigma_{k_0}^2}, \quad (26)$$

and the approximation has been made under the assumption of low energy s -wave scattering. Then by (21), (22) and (25) we find

$$\begin{aligned} \varepsilon\psi_{\text{scat}}(\mathbf{k}_1, \mathbf{k}_2, t) &\approx \left(\frac{1}{2\pi\sigma_{k_0}^2} \right)^{3/2} \exp \left[-\frac{\mathbf{K}^2 + 4(\mathbf{k} - k_0\hat{\mathbf{k}})^2}{8\sigma_{k_0}^2} \right] \\ &\times \varrho(k) e^{-i(k-k_0)R_0 - i\hbar K^2 t/(2M) - i\hbar k^2 t/(2\mu)}. \end{aligned} \quad (27)$$

Due to the fact that

$$\begin{aligned} &\exp \left[-\frac{(\mathbf{k}_1 - \mathbf{k}_0)^2 + (\mathbf{k}_2 + \mathbf{k}_0)^2}{4\sigma_{k_0}^2} \right] e^{-i(k_1-k_0)R_0/2 + i(k_2+k_0)R_0/2 - i\hbar(k_1^2+k_2^2)t/(2m)} \\ &= \exp \left[-\frac{\mathbf{K}^2 + 4(\mathbf{k} - k_0\hat{\mathbf{k}})^2}{8\sigma_{k_0}^2} \right] e^{-i(k-k_0)R_0 - i\hbar K^2 t/(2M) - i\hbar k^2 t/(2\mu)}, \end{aligned} \quad (28)$$

we may combine (20) and (27) to rewrite (19) as

$$\begin{aligned} \psi(\mathbf{k}_1, \mathbf{k}_2, t) &= (N)^{-1/2} \left(\frac{1}{2\pi\sigma_{k_0}^2} \right)^{3/2} \exp \left[-\frac{\mathbf{K}^2 + 4(\mathbf{k} - k_0\hat{\mathbf{k}})^2}{8\sigma_{k_0}^2} \right] \\ &\times [1 + \varrho(k)] e^{-i(k-k_0)R_0 - i\hbar K^2 t/(2M) - i\hbar k^2 t/(2\mu)}. \end{aligned} \quad (29)$$

Our dimensional analysis carried out in the separable case (see the previous Subsection) applies equally well in the entangled case. Hence, we may reduce the three-dimensional wave-packet $\psi(\mathbf{k}_1, \mathbf{k}_2, t)$ expressed via (19) to the one-dimensional one,

$$\psi(k_1, k_2, t) = (N)^{-1/2} \left[\psi_1(k_1) \psi_2(k_2) e^{-i\hbar(k_1^2+k_2^2)t/(2m)} + \varepsilon\psi_{\text{scat}}(k_1, k_2, t) \right], \quad (30)$$

where the single-particle wave function $\psi_{1/2}(k_{1/2})$ is specified via (16) and (17), together with $k_{1/2} = \frac{p_{1/2}}{\hbar} \in (-\infty, +\infty)$, $\langle k_{1/2} \rangle_o = \frac{\langle p_{1/2} \rangle_o}{\hbar} = \pm \frac{p_o}{\hbar} = \pm k_o$, $\sigma_{k_o} = \frac{\sigma_o}{\hbar}$, $q_{1/2} = \mp \frac{1}{2} R_o$. In analogy to (29), $\psi(k_1, k_2, t)$ can be rewritten as

$$\begin{aligned} \psi(k_1, k_2, t) &= (N)^{-1/2} \left(\frac{1}{2\pi\sigma_{k_o}^2} \right)^{1/2} \exp \left[-\frac{K^2 + 4(k - k_o)^2}{8\sigma_{k_o}^2} \right] \\ &\times [1 + \varrho(k)] e^{-i(k - k_o)R_o - i\hbar K^2 t/(2M) - i\hbar k^2 t/(2\mu)}, \end{aligned} \quad (31)$$

where we adopt the one-dimensional center of mass and relative coordinates, whose conjugate momenta are defined as $K \equiv k_1 + k_2 \in (-\infty, +\infty)$ and $k \equiv \frac{1}{2}(k_1 - k_2) \in (-\infty, +\infty)$, and $\varrho(k)$ is given by (26). Separating variables in (31), we may write

$$\begin{aligned} |\psi(k_1, k_2, t)|^2 &= \psi(k_1, k_2, t) \psi^*(k_1, k_2, t) \\ &= \frac{N^{-1}}{2\pi\sigma_{k_o}^2} \exp \left(-\frac{K^2}{4\sigma_{k_o}^2} \right) \exp \left(-\frac{\tilde{k}^2}{\sigma_{k_o}^2} \right) \left[1 + 2\Re(\varrho(k)) + |\varrho(k)|^2 \right], \end{aligned} \quad (32)$$

where $\tilde{k} \equiv k - k_o$ and \Re denotes the real part of $\varrho(k)$. However, we find that the complex-valued scattering amplitude $f(k)$ can be approximated as real since $f(k) = \frac{\theta(k)}{k} + \mathcal{O}(\theta^2)$ for $\theta(k) \ll 1$. In view of this fact and (26) we employ the following approximations:

$$\Re(\varrho(k)) \approx 4R_o k^2 f(k), \quad (33)$$

$$|\varrho(k)|^2 \approx \frac{16(k_o^2 + \sigma_{k_o}^4 R_o^2) k^4 (f(k))^2}{\sigma_{k_o}^4}. \quad (34)$$

Upon Taylor expanding $f(k)$ and $(f(k))^2$ around $k = k_o$, we re-express $\Re(\varrho(k))$ in (33) and $|\varrho(k)|^2$ in (34) as

$$\Re(\varrho(k)) \approx 4R_o k^2 \sum_{n=0}^{\infty} A_n (k - k_o)^n \quad (35)$$

and

$$|\varrho(k)|^2 \approx \frac{16(k_o^2 + \sigma_{k_o}^4 R_o^2) k^4}{\sigma_{k_o}^4} \sum_{n=0}^{\infty} B_n (k - k_o)^n, \quad (36)$$

respectively. Here the quantities A_n and B_n are appropriate coefficients determined from the expansions $f(k) = \sum_{n=0}^{\infty} A_n (k - k_o)^n$ and $(f(k))^2 = \sum_{n=0}^{\infty} B_n (k - k_o)^n$, respectively. By inserting (35) and (36) into (32) we can integrate $|\psi|^2$ as follows:

$$\begin{aligned} 2\pi\sigma_{k_o}^2 N \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\psi|^2 dk_1 dk_2 &= 2\pi\sigma_{k_o}^2 N \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\psi|^2 dK d\tilde{k} \\ &\approx \int_{-\infty}^{+\infty} dK \exp \left(-\frac{K^2}{4\sigma_{k_o}^2} \right) \int_{-\infty}^{+\infty} d\tilde{k} \exp \left(-\frac{\tilde{k}^2}{\sigma_{k_o}^2} \right) \left[1 + 8R_o \left(\tilde{k}^2 + 2k_o \tilde{k} + k_o^2 \right) \sum_{n=0}^{\infty} A_n \tilde{k}^n + \right. \\ &\quad \left. + \frac{16(k_o^2 + \sigma_{k_o}^4 R_o^2)}{\sigma_{k_o}^4} \left(\tilde{k}^4 + 4k_o \tilde{k}^3 + 6k_o^2 \tilde{k}^2 + 4k_o^3 \tilde{k} + k_o^4 \right) \sum_{n=0}^{\infty} B_n \tilde{k}^n \right], \end{aligned} \quad (37)$$

where $\tilde{k} \equiv k - k_o$. From [43] we find

$$\int_{-\infty}^{+\infty} d\tilde{k} \exp \left(-\frac{\tilde{k}^2}{\sigma_{k_o}^2} \right) \tilde{k}^n = \delta_{n,2m} (2m-1)!! \sqrt{\pi} \sigma_{k_o} \left(\frac{\sigma_{k_o}^2}{2} \right)^m, \quad (38)$$

where $m = 0, 1, 2, \dots$. It should be noted however, that we have approximated $f(k)$ as a real-valued function $f(k) \approx \frac{\theta(k)}{k}$, assuming $\theta(k) \ll 1$. For low energy s -wave scattering, which is the case presently under consideration, we have $k \ll 1$ and $\theta(k) = -ka_s + \mathcal{O}(k^2)$, where the parameter a_s of dimension length is defined as the s -wave scattering length [44]. This leads to $f(k) \approx -a_s$ and $f^{[p]}(k) \approx 0$, where the superscript $^{[p]}$ denotes any p -th order derivative ($p = 1, 2, \dots$). Hence, we have $A_0 = -a_s$, $A_1 = A_2 = \dots = 0$ and $B_0 = a_s^2$, $B_1 = B_2 = \dots = 0$. Making use of these coefficients as well as (38), one may compute the integral in (37) to obtain

$$\begin{aligned} 2\pi\sigma_{ko}^2 N \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\psi|^2 dk_1 dk_2 &= 2\pi\sigma_{ko}^2 N \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\psi|^2 dK d\tilde{k} \\ &\approx 2\pi\sigma_{ko}^2 \left[1 - 4(2k_o^2 + \sigma_{ko}^2) R_o a_s + \frac{4(k_o^2 + \sigma_{ko}^4 R_o^2)(4k_o^4 + 12k_o^2 \sigma_{ko}^2 + 3\sigma_{ko}^4)}{\sigma_{ko}^4} a_s^2 \right]. \end{aligned} \quad (39)$$

Indeed, one would obtain the same result as (39) to the leading order by evaluating the following integral:

$$\begin{aligned} &\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk_1 dk_2 \exp \left\{ -\frac{1}{2(1-r_{QM}^2)} \left[\frac{(k_1 - k_o)^2}{\sigma_{ko}^2} - 2r_{QM} \frac{(k_1 - k_o)(k_2 + k_o)}{\sigma_{ko}^2} + \frac{(k_2 + k_o)^2}{\sigma_{ko}^2} \right] \right\} \\ &= 2\pi\sigma_{ko}^2 \sqrt{1 - r_{QM}^2} = 2\pi\sigma_{ko}^2 \left[1 - \frac{1}{2}r_{QM}^2 + \mathcal{O}(r_{QM}^4) \right], \end{aligned} \quad (40)$$

where

$$r_{QM} \equiv \sqrt{8(2k_o^2 + \sigma_{ko}^2) R_o a_s} \ll 1. \quad (41)$$

Therefore, to a good approximation, we may replace the probability density in (32) with

$$P_{QM}^{\text{after}} \equiv \left| \psi^{\text{after}}(k_1, k_2, t) \right|^2 \simeq \frac{\exp \left\{ -\frac{1}{2(1-r_{QM}^2)} \left[\frac{(k_1 - k_o)^2}{\sigma_{ko}^2} - 2r_{QM} \frac{(k_1 - k_o)(k_2 + k_o)}{\sigma_{ko}^2} + \frac{(k_2 + k_o)^2}{\sigma_{ko}^2} \right] \right\}}{2\pi\sigma_{ko}^2 \sqrt{1 - r_{QM}^2}}, \quad (42)$$

where the integral in (40) has been normalized. In case $r_{QM} = 0$, (42) reduces to

$$P_{QM}^{\text{after}} = P_{QM}^{\text{before}} = \left| \psi^{\text{before}}(k_1, k_2) \right|^2 = \frac{1}{2\pi\sigma_{ko}^2} \exp \left[-\frac{(k_1 - k_o)^2 + (k_2 + k_o)^2}{2\sigma_{ko}^2} \right], \quad (43)$$

which is verified via (15), (16) and (17).

The obtained expressions for the probability densities P_{QM}^{before} and P_{QM}^{after} motivate our information geometric investigation as will be explained in the next Section.

III. THE INFORMATION GEOMETRIC PERSPECTIVE

In this Section, we outline the main ideas behind the IGAC and present the information geometry of correlated and uncorrelated Gaussian statistical manifolds employed in our investigation of scattering induced quantum entanglement.

A. On the IGAC

IGAC [45, 46] is a theoretical framework developed to study the complexity of informational geodesic flows describing physical. It is the information geometric analogue of conventional geometrodynamical approaches to chaos [29, 30, 47–49] where the classical configuration space is replaced by a curved statistical manifold with the additional possibility of considering chaotic dynamics arising from non-conformally flat metrics. Additionally, it is an information geometric extension of the Jacobi geometrodynamics (the geometrization of a Hamiltonian system by transforming it to a geodesic flow [50]).

More specifically, IGAC is the application of entropic dynamics (ED) [51] to complex systems of arbitrary nature. ED is a theoretical framework that arises from the combination of inductive inference (Maximum Relative Entropy methods, [52, 53]) and Information Geometry (IG), that is, Riemannian geometry applied to probability theory [54]. IGAC extends the applicability of ED to temporally-complex (chaotic) dynamical systems on curved statistical manifolds and relevant measures of chaoticity of such an IGAC have been identified [45].

The essential ideas underlying the IGAC and the construction of statistical manifolds are presented in what follows. Let the probability distribution function (PDF) $P(X|\Theta)$ represent the maximally probable description of the system being considered. The quantity X is a random variable that represents a microstate of the system, while Θ represents a macrostate. The sets $\{X\}$ and $\{\Theta\}$ form the microspace \mathcal{X} and the parameter space \mathcal{D}_Θ , respectively. The set of probability distributions forms the statistical manifold \mathcal{M} . A geodesic curve on a curved statistical manifold \mathcal{M} represents the maximum probability path a complex dynamical system explores in its evolution between initial and final macrostates Θ_I and Θ_F , respectively. Each point of the geodesic on an n -dimensional statistical manifold \mathcal{M} represents a macrostate parametrized by the macroscopic dynamical variables $\{\Theta\}$. Furthermore, each macrostate is in a one-to-one correspondence with the probability distribution $P(X|\Theta)$ representing the maximally probable description of the system being considered. The main goal of an ED model is that of inferring “*macroscopic predictions*” in the absence of detailed knowledge of the microscopic nature of the arbitrary complex systems being considered. More explicitly, by “macroscopic prediction” we mean knowledge of the statistical parameters (expectation values) of the probability distribution function that best reflects what is known about the system. This is an important conceptual point. The probability distribution reflects the system in general, not the microstates. Once the microstates have been defined, we then select the relevant information about the system. In other words, we have to select the macrospace of the system. We emphasize that knowledge of both initial and final macrostates is not necessary to carry out macroscopic predictions. For instance, one may only have knowledge of the initial state and assume that the system evolves to other states, without actually knowing what the final state is. In such a case, it can be shown that the system moves continuously and irreversibly along the entropy gradient [55]. We note that in its present form the IGAC can only be applied to CVQS. This restricted applicability is due to the fact that the IGAC is used to understand the evolution of continuous trajectories on \mathcal{M} . In the context of quantum mechanical systems, the set $\{\Theta\}$ would correspond to the continuous eigenvalue spectrum of some observable (expected values). The IGAC must be reformulated in order to be applicable to general quantum systems. Such a reformulation is currently in progress.

For a brief overview of some of the latest applications of the IGAC to both classical and quantum scenarios, we refer to [56].

B. Gaussian Statistical Models and Micro-correlations

Here, we introduce the notion of Gaussian statistical models (manifolds) in either absence or presence of correlations between the microscopic degrees of freedom of the system (i.e. micro-correlations).

1. Statistical Models in Absence of Micro-correlations

Consider a statistical model whose microstates span a n -dimensional space labeled by the variables $\{X\} = \{x_1, x_2, \dots, x_n\}$ with $x_j \in \mathbb{R}$, $\forall j = 1, \dots, n$. We assume the only testable information pertaining to the quantities x_j consists of the expectation values $\langle x_j \rangle$ and the variance Δx_j . The set of these expected values define the $2n$ -dimensional space of macrostates of the system. Our $2n$ -dimensional statistical model represents a macroscopic (i.e. probabilistic) description of a microscopic, n -dimensional physical system evolving over a n -dimensional (micro) space. We assume that all information relevant to the dynamical evolution of the system is contained in the probability distributions. For this reason, no other information is required. Each macrostate may be thought as a point of a $2n$ -dimensional statistical manifold with coordinates given by the numerical values of the expectations $^{(1)}\vartheta_j$ and $^{(2)}\vartheta_j$. The available *relevant information* can be written in the form of the following $2n$ information constraint equations:

$$\langle x_j \rangle = \int_{-\infty}^{+\infty} dx_j x_j P_j \left(x_j | ^{(1)}\vartheta_j, ^{(2)}\vartheta_j \right), \quad \Delta x_j = \left[\int_{-\infty}^{+\infty} dx_j (x_j - \langle x_j \rangle)^2 P_j \left(x_j | ^{(1)}\vartheta_j, ^{(2)}\vartheta_j \right) \right]^{\frac{1}{2}}. \quad (44)$$

The probability distributions $P_j \left(x_j | ^{(1)}\vartheta_j, ^{(2)}\vartheta_j \right)$ in (44) are constrained by the conditions of normalization,

$$\int_{-\infty}^{+\infty} dx_j P_j \left(x_j | ^{(1)}\vartheta_j, ^{(2)}\vartheta_j \right) = 1. \quad (45)$$

Maximum Relative Entropy methods [52, 53, 57, 58] allow us to associate a probability distribution $P(X|\Theta)$ to each point in the space of states $\{\Theta\}$. The distribution that best reflects the information contained in the prior distribution $m(X)$ updated by the information $(\langle x_j \rangle, \Delta x_j)$ is obtained by maximizing the relative entropy

$$S(\Theta) = - \int dX P(X|\Theta) \ln \left(\frac{P(X|\Theta)}{m(X)} \right). \quad (46)$$

As a working hypothesis, the prior $m(X)$ is set to be uniform since we assume the lack of prior available information about the system (postulate of equal *a priori* probabilities). Information theory identifies the Gaussian distribution as the maximum entropy distribution if only the expectation value and the variance are known [59]. Indeed, upon maximizing (46) given the constraints (44) and (45), we obtain

$$P(X|\Theta) = \prod_{j=1}^n P_j \left(x_j | {}^{(1)}\vartheta_j, {}^{(2)}\vartheta_j \right), \quad (47)$$

where

$$P_j \left(x_j | {}^{(1)}\vartheta_j, {}^{(2)}\vartheta_j \right) = (2\pi\sigma_j^2)^{-\frac{1}{2}} \exp \left[-\frac{(x_j - \mu_j)^2}{2\sigma_j^2} \right], \quad (48)$$

and in standard notation for Gaussians, ${}^{(1)}\vartheta_j \stackrel{\text{def}}{=} \langle x_j \rangle \equiv \mu_j$, ${}^{(2)}\vartheta_j \stackrel{\text{def}}{=} \Delta x_j \equiv \sigma_j$. The probability distribution (47) encodes the available information concerning the system.

The statistical manifold \mathcal{M} associated to (47) is formally defined as follows:

$$\mathcal{M} = \left\{ P(X|\Theta) = \prod_{j=1}^n P_j(x_j | \mu_j, \sigma_j) : \Theta = (\vartheta^1, \dots, \vartheta^{2n}) \in \mathcal{D}_{\Theta}^{(\text{total})} \mid P(X|\Theta) \geq 0 \right\}. \quad (49)$$

The parameter space $\mathcal{D}_{\Theta}^{(\text{total})}$ (homeomorphic to \mathcal{M}) is defined as

$$\mathcal{D}_{\Theta}^{(\text{total})} \stackrel{\text{def}}{=} \bigotimes_{k=1}^{2n} \mathcal{I}_{\vartheta^k} = (\mathcal{I}_{\vartheta^1} \otimes \mathcal{I}_{\vartheta^2} \cdots \otimes \mathcal{I}_{\vartheta^{2n}}) \subseteq \mathbb{R}^{2n}, \quad (50)$$

where $\mathcal{I}_{\vartheta^k}$ is a subset of \mathbb{R} and represents the entire range of accessible values for the macrovariable ϑ^k .

The line element ds^2 arising from (47) is [60]

$$ds_{\mathcal{M}}^2 = g_{ab}(\Theta) d\vartheta^a d\vartheta^b = \sum_{j=1}^n \left(\frac{1}{\sigma_j^2} d\mu_j^2 + \frac{2}{\sigma_j^2} d\sigma_j^2 \right) \text{ with } a, b = 1, \dots, 2n. \quad (51)$$

Note that we have assumed uncoupled constraints among microvariables x_j . In other words, we assumed that information about correlations between the microvariables need not to be tracked. This assumption leads to the simplified product rule (47).

A measure of distinguishability among the macrostates of the Gaussian model is achieved by assigning a probability distribution $P(X|\Theta)$ to each $2n$ -dimensional macrostate $\Theta \stackrel{\text{def}}{=} \{({}^{(1)}\vartheta_j, {}^{(2)}\vartheta_j)\}_{n\text{-pairs}} = \{(\langle x_j \rangle, \Delta x_j)\}_{n\text{-pairs}}$. The process of assigning a probability distribution to each state provides \mathcal{M} with a metric structure. Specifically, the Fisher-Rao information metric $g_{ab}(\Theta)$ [54] is a measure of distinguishability among macrostates on the statistical manifold \mathcal{M} ,

$$g_{ab}(\Theta) = \int dX P(X|\Theta) \partial_a \ln P(X|\Theta) \partial_b \ln P(X|\Theta) = 4 \int dX \partial_a \sqrt{P(X|\Theta)} \partial_b \sqrt{P(X|\Theta)}, \quad (52)$$

with $a, b = 1, \dots, 2n$ and $\partial_a = \frac{\partial}{\partial \vartheta^a}$. It assigns an information geometry to the space of states. The information metric $g_{ab}(\Theta)$ is a symmetric and positive definite Riemannian metric. For the sake of completeness and in view of its potential relevance in the study of correlations, we point out that the Fisher-Rao metric satisfies the following two properties: 1) invariance under (invertible) transformations of microvariables $\{X\} \in \mathcal{X}$; 2) covariance under

reparametrization of the statistical macrospace $\{\Theta\} \in \mathcal{D}_\Theta$. The invariance of $g_{ab}(\Theta)$ under reparametrization of the microspace \mathcal{X} implies that [54],

$$\mathcal{X} \subseteq \mathbb{R}^n \ni x \mapsto y \stackrel{\text{def}}{=} f(x) \in \mathcal{Y} \subseteq \mathbb{R}^n \implies p(x|\vartheta) \mapsto p'(y|\vartheta') = \left[\frac{1}{\left| \frac{\partial f}{\partial x} \right|} p(x|\vartheta) \right]_{x=f^{-1}(y)}. \quad (53)$$

The covariance under reparametrization of the parameter space \mathcal{D}_Θ (homeomorphic to \mathcal{M}) implies that [54],

$$\mathcal{D}_\Theta \ni \vartheta \mapsto \vartheta' \stackrel{\text{def}}{=} f(\vartheta) \in \mathcal{D}_{\Theta'} \implies g_{ab}(\vartheta) \mapsto g'_{ab}(\vartheta') = \left[\frac{\partial \vartheta^c}{\partial \vartheta'^a} \frac{\partial \vartheta^d}{\partial \vartheta'^b} g_{cd}(\vartheta) \right]_{\vartheta=f^{-1}(\vartheta')}, \quad (54)$$

where

$$g'_{ab}(\vartheta') = \int dx p'(x|\vartheta') \partial'_a \ln p'(x|\vartheta') \partial'_b \ln p'(x|\vartheta'), \quad (55)$$

with $\partial'_a = \frac{\partial}{\partial \vartheta'^a}$ and $p'(x|\vartheta') = p(x|\vartheta = f^{-1}(\vartheta'))$.

2. Statistical Models in Presence of Micro-correlations

Coupled constraints would lead to a “generalized” product rule in (47) and to a metric tensor (52) with non-trivial off-diagonal elements (covariance terms). In presence of correlated degrees of freedom $\{x_j\}$, the “generalized” product rule becomes

$$P_{\text{total}}(x_1, \dots, x_n) = \prod_{j=1}^n P_j(x_j) \xrightarrow{\text{correlations}} P'_{\text{total}}(x_1, \dots, x_n) \neq \prod_{j=1}^n P_j(x_j), \quad (56)$$

where

$$P'_{\text{total}}(x_1, \dots, x_n) = P_n(x_n|x_1, \dots, x_{n-1}) P_{n-1}(x_{n-1}|x_1, \dots, x_{n-2}) \cdots P_2(x_2|x_1) P_1(x_1). \quad (57)$$

For instance, correlations in the degrees of freedom may be introduced in terms of the following information-constraints,

$$x_j = f_j(x_1, \dots, x_{j-1}), \quad \forall j = 2, \dots, n. \quad (58)$$

In such a case, we obtain

$$P'_{\text{total}}(x_1, \dots, x_n) = \delta(x_n - f_n(x_1, \dots, x_{n-1})) \delta(x_{n-1} - f_{n-1}(x_1, \dots, x_{n-2})) \cdots \delta(x_2 - f_2(x_1)) P_1(x_1), \quad (59)$$

where the j -th probability distribution $P_j(x_j)$ is given by

$$P_j(x_j) = \int \cdots \int dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_n P'_{\text{total}}(x_1, \dots, x_n). \quad (60)$$

Correlations between the microscopic degrees of freedom of the system $\{x_j\}$ (micro-correlations) are conventionally introduced by means of the correlation coefficients $r_{ij}^{(\text{micro})}$ [61],

$$r_{ij}^{(\text{micro})} = r(x_i, x_j) \stackrel{\text{def}}{=} \frac{\langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle}{\sigma_i \sigma_j} \quad \text{with } \sigma_i = \sqrt{\langle (x_i - \langle x_i \rangle)^2 \rangle}, \quad (61)$$

with $r_{ij}^{(\text{micro})} \in (-1, 1)$ and $i, j = 1, \dots, n$. For the $2n$ -dimensional Gaussian statistical model in presence of micro-correlations, the system is described by the following probability distribution $P(X|\Theta)$:

$$P(X|\Theta) = \frac{1}{[(2\pi)^n \det C(\Theta)]^{\frac{1}{2}}} \exp \left[-\frac{1}{2} (X - M)^t \cdot C^{-1}(\Theta) \cdot (X - M) \right] \neq \prod_{j=1}^n (2\pi\sigma_j^2)^{-\frac{1}{2}} \exp \left[-\frac{(x_j - \mu_j)^2}{2\sigma_j^2} \right], \quad (62)$$

where $X = (x_1, \dots, x_n)$, $M = (\mu_1, \dots, \mu_n)$ and $C(\Theta)$ is the $(2n \times 2n)$ -dimensional (non-singular) covariance matrix.

In what follows, we will introduce the three-dimensional micro-correlated Gaussian statistical model being investigated.

C. The Two-variable Micro-correlated Gaussian Statistical Model

Consider micro-correlated Gaussian statistical models with $2n = 4$. For $n = 2$, (62) leads to the probability distribution $P(x, y | \mu_x, \mu_y, \sigma_x, \sigma_y)$ [61],

$$P(x, y | \mu_x, \mu_y, \sigma_x, \sigma_y; r) = \frac{\exp \left\{ -\frac{1}{2(1-r^2)} \left[\frac{(x-\mu_x)^2}{\sigma_x^2} - 2r \frac{(x-\mu_x)(y-\mu_y)}{\sigma_x \sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2} \right] \right\}}{2\pi \sigma_x \sigma_y \sqrt{1-r^2}}, \quad (63)$$

a bivariate normal distribution where $\sigma_x > 0$, $\sigma_y > 0$, $r \in (-1, 1)$, $X = (x, y)$, $\Theta = (\mu_x, \mu_y, \sigma_x, \sigma_y)$ and $C(\Theta)$,

$$C_{ij} = \begin{bmatrix} \sigma_x^2 & r\sigma_x\sigma_y + \mu_x\mu_y \\ r\sigma_y\sigma_x + \mu_y\mu_x & \sigma_y^2 \end{bmatrix} \text{ with } i, j = 1, 2. \quad (64)$$

Substituting (63) in (52), the Fisher-Rao information metric $g_{ab}(\mu_x, \mu_y, \sigma_x, \sigma_y; r)$ becomes

$$g_{ab}(\mu_x, \mu_y, \sigma_x, \sigma_y; r) = \begin{pmatrix} -\frac{1}{\sigma_x^2(r^2-1)} & 0 & \frac{r}{\sigma_x\sigma_y(r^2-1)} & 0 \\ 0 & -\frac{2-r^2}{\sigma_x^2(r^2-1)} & 0 & \frac{r^2}{\sigma_x\sigma_y(r^2-1)} \\ \frac{r}{\sigma_x\sigma_y(r^2-1)} & 0 & -\frac{1}{\sigma_y^2(r^2-1)} & 0 \\ 0 & \frac{r^2}{\sigma_x\sigma_y(r^2-1)} & 0 & -\frac{2-r^2}{\sigma_y^2(r^2-1)} \end{pmatrix}. \quad (65)$$

The infinitesimal line element $ds^2_{\mathcal{M}_{\text{corr}}^{4D}}$ relative to $g_{ab}(\mu_x, \mu_y, \sigma_x, \sigma_y; r)$ is given by

$$ds^2_{\mathcal{M}_{\text{corr}}^{4D}} = g_{11}(\sigma_x; r) d\mu_x^2 + g_{33}(\sigma_y; r) d\mu_y^2 + g_{22}(\sigma_x; r) d\sigma_x^2 + g_{44}(\sigma_y; r) d\sigma_y^2 + 2g_{13}(\sigma_x, \sigma_y; r) d\mu_x d\mu_y + 2g_{24}(\sigma_x, \sigma_y; r) d\sigma_x d\sigma_y, \quad (66)$$

where

$$\begin{aligned} g_{11}(\sigma_x; r) &= -\frac{1}{\sigma_x^2(r^2-1)}, \quad g_{13}(\sigma_x, \sigma_y; r) = \frac{r}{\sigma_x\sigma_y(r^2-1)}, \quad g_{22}(\sigma_x; r) = -\frac{2-r^2}{\sigma_x^2(r^2-1)}, \\ g_{24}(\sigma_x, \sigma_y; r) &= \frac{r^2}{\sigma_x\sigma_y(r^2-1)}, \quad g_{31}(\sigma_x, \sigma_y; r) = \frac{r}{\sigma_x\sigma_y(r^2-1)}, \quad g_{33}(\sigma_y; r) = -\frac{1}{\sigma_y^2(r^2-1)}, \\ g_{42}(\sigma_x, \sigma_y; r) &= \frac{r^2}{\sigma_x\sigma_y(r^2-1)}, \quad g_{44}(\sigma_y; r) = -\frac{2-r^2}{\sigma_y^2(r^2-1)}. \end{aligned} \quad (67)$$

It is rather difficult to present an analytical study of the IGAC associated with infinitesimal line element $ds^2_{\mathcal{M}_{\text{corr}}^{4D}}$ in (66). Such a study will be the subject of forthcoming investigations. In the present work we consider a special class of Gaussian models, namely those in which $\sigma_y = \sigma_x = \sigma$. Then the probability distribution $P(x, y | \mu_x, \mu_y, \sigma_x, \sigma_y; r)$ in (63) can be reduced to a simpler form,

$$P(x, y | \mu_x, \mu_y, \sigma; r) = \frac{\exp \left\{ -\frac{1}{2(1-r^2)} \left[\frac{(x-\mu_x)^2}{\sigma^2} - 2r \frac{(x-\mu_x)(y-\mu_y)}{\sigma^2} + \frac{(y-\mu_y)^2}{\sigma^2} \right] \right\}}{2\pi \sigma^2 \sqrt{1-r^2}}, \quad (68)$$

where $\sigma > 0$, $X = (x, y)$, $\Theta = (\mu_x, \mu_y, \sigma)$ and $C(\Theta)$,

$$C_{ij} = \begin{bmatrix} \sigma^2 & r\sigma^2 + \mu_x\mu_y \\ r\sigma^2 + \mu_y\mu_x & \sigma^2 \end{bmatrix} \text{ with } i, j = 1, 2. \quad (69)$$

The Fisher-Rao matrix $g_{ab}(\mu_x, \mu_y, \sigma; r)$ associated with $P(x, y | \mu_x, \mu_y, \sigma; r)$ reads

$$g_{ab}(\mu_x, \mu_y, \sigma; r) = \frac{1}{\sigma^2} \begin{pmatrix} -\frac{1}{r^2-1} & \frac{r}{r^2-1} & 0 \\ \frac{r}{r^2-1} & -\frac{1}{r^2-1} & 0 \\ 0 & 0 & 4 \end{pmatrix}. \quad (70)$$

The line element associated with metric $g_{ab}(\mu_x, \mu_y, \sigma; r)$ is given by

$$\begin{aligned} ds_{\mathcal{M}_{\text{corr}}^{\text{3D}}}^2 &= g_{11}(\sigma_x; r) d\mu_x^2 + g_{33}(\sigma_y; r) d\mu_y^2 + 2g_{13}(\sigma; r) d\mu_x d\mu_y + [g_{22}(\sigma; r) + g_{44}(\sigma; r) + 2g_{24}(\sigma; r)] d\sigma^2 \\ &= \frac{1}{\sigma^2} \left(\frac{1}{1-r^2} d\mu_x^2 + \frac{1}{1-r^2} d\mu_y^2 - \frac{2r}{1-r^2} d\mu_x d\mu_y + 4d\sigma^2 \right). \end{aligned} \quad (71)$$

Observe that in the absence of micro-correlations, the two-variable probability distribution (68) reduces to

$$P(x, y | \mu_x, \mu_y, \sigma) = \frac{1}{2\pi\sigma^2} \exp \left[-\frac{(x - \mu_x)^2}{2\sigma^2} \right] \exp \left[-\frac{(y - \mu_y)^2}{2\sigma^2} \right], \quad (72)$$

while the metric and corresponding line element become

$$g_{ab}(\mu_x, \mu_y, \sigma) = \frac{1}{\sigma^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad (73)$$

and

$$ds_{\mathcal{M}_{\text{non-corr}}^{\text{3D}}}^2 = \frac{1}{\sigma^2} (d\mu_x^2 + d\mu_y^2 + 4d\sigma^2), \quad (74)$$

respectively. In what follows we limit our analysis to the study of non-negative micro-correlations, that is, we will consider $r \in [0, 1)$.

D. The Information Dynamics on the Statistical Manifold $\mathcal{M}_{\text{corr}}^{\text{3D}}$.

The information dynamics on the manifold $\mathcal{M}_{\text{corr}}^{\text{3D}}$ represented by (70) can be derived from a standard principle of least action of Jacobi type [51]. The geodesic equations for the macrovariables of the Gaussian ED model are given by *nonlinear* second order coupled ordinary differential equations,

$$\frac{d^2 \vartheta^a}{d\tau^2} + \Gamma_{bc}^a \frac{d\vartheta^b}{d\tau} \frac{d\vartheta^c}{d\tau} = 0, \quad (75)$$

where $a, b, c = 1, 2, 3$ and we denote $\vartheta^1 = \mu_1 = \mu_x$, $\vartheta^2 = \mu_2 = \mu_y$, $\vartheta^3 = \sigma$. The connection coefficients Γ_{bc}^a appearing in (75) are defined as [62]

$$\Gamma_{bc}^a = \frac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}). \quad (76)$$

In our case, through (70) the non-vanishing connection coefficients are given by

$$\Gamma_{13}^1 = -\frac{1}{\sigma}, \quad \Gamma_{23}^2 = -\frac{1}{\sigma} = \Gamma_{32}^2, \quad \Gamma_{11}^3 = -\frac{1}{4\sigma(r^2 - 1)}, \quad \Gamma_{12}^3 = \frac{r}{4\sigma(r^2 - 1)} = \Gamma_{21}^3, \quad \Gamma_{22}^3 = -\frac{1}{4\sigma(r^2 - 1)}, \quad \Gamma_{33}^3 = -\frac{1}{\sigma}. \quad (77)$$

The geodesic equations in (75) describe a *reversible* dynamics whose solution is the trajectory between an initial Θ_I and a final macrostate Θ_F . The trajectory can be equally well traversed in both directions. In the case under consideration, substituting (77) in (75), the three geodesic equations become

$$0 = \frac{d^2 \mu_1(\tau)}{d\tau^2} - \frac{2}{\sigma(\tau)} \frac{d\mu_1(\tau)}{d\tau} \frac{d\sigma(\tau)}{d\tau}, \quad (78)$$

$$0 = \frac{d^2 \mu_2(\tau)}{d\tau^2} - \frac{2}{\sigma(\tau)} \frac{d\mu_2(\tau)}{d\tau} \frac{d\sigma(\tau)}{d\tau}, \quad (79)$$

$$\begin{aligned} 0 &= \frac{d^2 \sigma(\tau)}{d\tau^2} - \frac{1}{\sigma(\tau)} \left(\frac{d\sigma(\tau)}{d\tau} \right)^2 - \frac{1}{4\sigma(\tau)(r^2 - 1)} \left[\left(\frac{d\mu_1(\tau)}{d\tau} \right)^2 + \left(\frac{d\mu_2(\tau)}{d\tau} \right)^2 \right] + \\ &\quad + \frac{r}{2\sigma(\tau)(r^2 - 1)} \frac{d\mu_1(\tau)}{d\tau} \frac{d\mu_2(\tau)}{d\tau}. \end{aligned} \quad (80)$$

Integration of the above coupled system of nonlinear differential equations is non-trivial. A detailed derivation of the geodesic paths is given in Appendix A. After integration of (78), (79) and (80), the geodesic trajectories for the non-correlated Gaussian system become,

$$\mu_1(\tau; 0) = -\sqrt{p_o^2 + 2\sigma_o^2} \tanh(A_o\tau), \quad (81)$$

$$\mu_2(\tau; 0) = \sqrt{p_o^2 + 2\sigma_o^2} \tanh(A_o\tau), \quad (82)$$

$$\sigma(\tau; 0) = \sqrt{\frac{1}{2}p_o^2 + \sigma_o^2} \frac{1}{\cosh(A_o\tau)}, \quad (83)$$

while for the correlated Gaussian system the geodesics read,

$$\mu_1(\tau; r) = -\sqrt{(1-r)(p_o^2 + 2\sigma_o^2)} \tanh(A_o\tau), \quad (84)$$

$$\mu_2(\tau; r) = \sqrt{(1-r)(p_o^2 + 2\sigma_o^2)} \tanh(A_o\tau), \quad (85)$$

$$\sigma(\tau; r) = \sqrt{\frac{1}{2}p_o^2 + \sigma_o^2} \frac{1}{\cosh(A_o\tau)}, \quad (86)$$

where the subscript “_o” denotes the initial state, and

$$\begin{aligned} A_o &\equiv \frac{1}{\tau_o} \sinh^{-1} \left(\frac{p_o}{\sqrt{2}\sigma_o} \right) \\ &\stackrel{\frac{\sigma_o}{p_o} \ll 1}{=} \frac{1}{\tau_o} \left\{ \ln \left(\frac{\sqrt{2}p_o}{\sigma_o} \right) + \frac{1}{2} \left(\frac{\sigma_o}{p_o} \right)^2 - \frac{3}{8} \left(\frac{\sigma_o}{p_o} \right)^4 + \mathcal{O} \left[\left(\frac{\sigma_o}{p_o} \right)^6 \right] \right\}, \end{aligned} \quad (87)$$

whose derivation is presented in Appendix B.

IV. APPLICATION OF INFORMATION GEOMETRY TO QUANTUM PHYSICS - PURITY, SCATTERING AND QUANTUM ENTANGLEMENT

In this Section we use information geometric techniques in conjunction with standard partial wave quantum scattering theory to provide an information geometric characterization of quantum entanglement.

A. Association of Quantum Systems with Information Geometric Systems

We now focus on applying IG methods to the quantum entanglement produced by a head-on collision between two Gaussian wave-packets in momentum space. We observe from (15) and (16) that the two-particle probability density before collision is given by $\left| \psi^{\text{before}}(k_1, k_2) \right|^2 = a^2(k_1, \langle k_1 \rangle_o; \sigma_{k_o}) a^2(k_2, \langle k_2 \rangle_o; \sigma_{k_o})$. By letting $x = p_1 = \hbar k_1$, $y = p_2 = \hbar k_2$, $\mu_x = \mu_{p_1} = \hbar \mu_{k_1}$, $\mu_y = \mu_{p_2} = \hbar \mu_{k_2}$, $\sigma = \sigma_p = \hbar \sigma_k$ in (72) and assigning $\mu_{p_1} \rightarrow \langle p_1 \rangle_o = \hbar \langle k_1 \rangle_o = \hbar k_o$, $\mu_{p_2} \rightarrow \langle p_2 \rangle_o = \hbar \langle k_2 \rangle_o = -\hbar k_o$, $\sigma_p \rightarrow \sigma_o = \hbar \sigma_{k_o}$, we can identify the non-correlated probability distribution (72) with $\left| \psi^{\text{before}}(k_1, k_2) \right|^2$ due to (43). That is,

$$P_{\text{QM}}^{\text{before}} = P_{\text{non-corr.}} \quad (88)$$

The information geometry associated with the two-particle system before collision is specified by metric (73).

In a similar manner, the probability density $\left| \psi^{\text{after}}(k_1, k_2, t) \right|^2$ in (42) is approximated with the Gaussian probability distribution (68). Comparison of (42) and (68) implies that when $r_{\text{QM}} \ll 1$,

$$P_{\text{QM}}^{\text{after}} \simeq P_{\text{corr.}}, \quad (89)$$

where

$$P_{\text{corr.}} \equiv \frac{\exp \left\{ -\frac{1}{2(1-r^2)} \left[\frac{(k_1 - k_o)^2}{\sigma_{k_o}^2} - 2r \frac{(k_1 - k_o)(k_2 + k_o)}{\sigma_{k_o}^2} + \frac{(k_2 + k_o)^2}{\sigma_{k_o}^2} \right] \right\}}{2\pi\sigma_{k_o}^2 \sqrt{1-r^2}}, \quad (90)$$

with

$$r = r_{\text{QM}}. \quad (91)$$

The expression of r_{QM} in terms of physical quantities is given by (41). The information geometry associated with the two-particle system after collision is specified by metric (70).

B. Purity as a Measure of Quantum Entanglement

The subsystem purity of a composite system of two particles engaged in a head-on collision was calculated in [24] by deriving the two-particle wave function modified by s -wave scattering amplitudes. They utilized the purity function \mathcal{P} as a measure of entanglement. Formally, the purity function is defined as

$$\mathcal{P} \stackrel{\text{def}}{=} \text{Tr}(\rho_1^2), \quad (92)$$

where $\rho_1 = \text{Tr}_2(\rho_{12})$ is the reduced density matrix of particle 1 and ρ_{12} is the two-particle density matrix associated with the post-collisional two-particle wave function, given by (19). For pure two-particle states, the smaller the value of \mathcal{P} the higher the entanglement. That is, the loss of purity provides an indicator of the degree of entanglement. Hence, a disentangled product state corresponds to $\mathcal{P} = 1$. We remark that although \mathcal{P} shares similar features to entropy, it is more readily accessible to theoretical analysis [12, 18, 63] than the latter. In atomic physics, \mathcal{P} has also been employed to quantify two-body correlations in a multitude of dynamical processes [12, 13, 18, 63].

Given the system, it was found in [24] that the purity is specifically expressed as

$$\mathcal{P} = \iiint \psi(\mathbf{k}_1, \mathbf{k}_2, t) \psi(\mathbf{k}_3, \mathbf{k}_4, t) \psi^*(\mathbf{k}_1, \mathbf{k}_4, t) \psi^*(\mathbf{k}_3, \mathbf{k}_2, t) d^3\mathbf{k}_1 d^3\mathbf{k}_2 d^3\mathbf{k}_3 d^3\mathbf{k}_4. \quad (93)$$

By employing the same dimensional analysis as developed in Section II, we may effectively reduce (93) to

$$\mathcal{P} = \iiint \psi(k_1, k_2, t) \psi(k_3, k_4, t) \psi^*(k_1, k_4, t) \psi^*(k_3, k_2, t) dk_1 dk_2 dk_3 dk_4. \quad (94)$$

Now, specifying the wave-packets in (94) by means of (26) and (31) and performing the integral, one obtains

$$\mathcal{P} = 1 - 8(2k_o^2 + \sigma_{ko}^2) R_o a_s + \mathcal{O}(a_s^2), \quad (95)$$

where the parameter a_s is the s -wave scattering length, defined from $f(k) \approx -a_s$ for $k \ll 1$. Employing the scattering cross section $\Sigma = 4\pi a_s^2$, we may express the purity in an alternative manner, namely,

$$\mathcal{P} = 1 - \frac{4(2k_o^2 + \sigma_{ko}^2) R_o \sqrt{\Sigma}}{\sqrt{\pi}} + \mathcal{O}(\Sigma). \quad (96)$$

The s -wave scattering can also be understood in terms of a scattering (interaction) potential and the scattering phase shift. Consider a scattering potential

$$V(x) = \begin{cases} V, & 0 \leq x \leq L \\ 0, & x > L \end{cases}, \quad (97)$$

where V denotes the height (for $V > 0$; repulsive potential) or depth (for $V < 0$; attractive potential) of the potential and L the range of the potential. Then solving the Schrödinger equation with this potential for the scattered wave, we are led to [64]

$$k_{\text{in}} \cot(k_{\text{in}} L) = k_{\text{out}} \cot(k_{\text{out}} L + \theta), \quad (98)$$

with

$$k_{\text{in}} = \frac{\sqrt{2\mu(\mathcal{E} - V)}}{\hbar}, \quad 0 < x < L, \quad (99)$$

$$k_{\text{out}} = \frac{\sqrt{2\mu\mathcal{E}}}{\hbar}, \quad x > L, \quad (100)$$

where μ and \mathcal{E} are the reduced mass and kinetic energy of the two-particle system in the relative coordinates, respectively, and k_{in} and k_{out} represent the conjugate-coordinate wave vectors inside and outside the potential region, respectively. Equation (98) together with (99) and (100) indicates that the scattering potential shifts the phase of the scattered wave by θ at points beyond the scattering region. In the next Subsection we will make use of this idea to determine the scattering phase shift which is linked with the micro-correlation coefficient r in our statistical model.

C. Information Geometric Interpretation of Quantum Entanglement

We utilize the results of our information dynamics given by equations (81), (82), (83), (84), (85) and (86) to furnish an information geometric interpretation of “quantum entanglement”, which is characterized by the purity given by (96).

1. Momentum-space Gaussian Statistical Models

To achieve the above task, one joins two different charts of Gaussian statistical manifolds, one without correlation (before collision) and the other with correlation (after collision). The two models can be represented by means of (43) and (90) with associated statistical manifolds (73) and (70), respectively.

The set of geodesic curves for each model is represented by equations (81), (82), (83) (for the non-correlated model) and by equations (84), (85), (86) (for the correlated model). The two sets are joined at the junction, $\tau = 0$: $\tau < 0$ (before collision) for the non-correlated model and $\tau \geq 0$ (after collision) for the correlated model.

The set of curves given by equations (81), (82), (83) may be assigned to $\{\langle p_{1b}(\tau) \rangle, \langle p_{2b}(\tau) \rangle, \langle \sigma_b(\tau) \rangle\}$ while the set given by equations (84), (85), (86) may be assigned to $\{\langle p_{1a}(\tau) \rangle, \langle p_{2a}(\tau) \rangle, \langle \sigma_a(\tau) \rangle\}$. The subscripts “1” and “2” denote particle 1 and particle 2, respectively; subscripts “b” and “a” denote ‘before’ and ‘after’ collision, respectively. Then we may write the following two sets of equations: for $\tau < 0$ (before collision),

$$\langle p_{1b}(\tau) \rangle = \mu_1(\tau; 0) = -\sqrt{p_o^2 + 2\sigma_o^2} \tanh(A_o \tau), \quad (101)$$

$$\langle p_{2b}(\tau) \rangle = \mu_2(\tau; 0) = \sqrt{p_o^2 + 2\sigma_o^2} \tanh(A_o \tau), \quad (102)$$

$$\langle \sigma_b(\tau) \rangle = \sigma(\tau; 0) = \sqrt{\frac{1}{2}p_o^2 + \sigma_o^2} \frac{1}{\cosh(A_o \tau)}, \quad (103)$$

while for $\tau \geq 0$ (after collision),

$$\langle p_{1a}(\tau) \rangle = \mu_1(\tau; r) = -\sqrt{(1-r)(p_o^2 + 2\sigma_o^2)} \tanh(A_o \tau), \quad (104)$$

$$\langle p_{2a}(\tau) \rangle = \mu_2(\tau; r) = \sqrt{(1-r)(p_o^2 + 2\sigma_o^2)} \tanh(A_o \tau), \quad (105)$$

$$\langle \sigma_a(\tau) \rangle = \sigma(\tau; r) = \sqrt{\frac{1}{2}p_o^2 + \sigma_o^2} \frac{1}{\cosh(A_o \tau)}, \quad (106)$$

where A_o is given by (87). Here we recognize that the momenta $\langle p_{1b}(\tau) \rangle$ and $\langle p_{1a}(\tau) \rangle$ asymptotically converge to $\sqrt{p_o^2 + 2\sigma_o^2}$ and $-\sqrt{(1-r)(p_o^2 + 2\sigma_o^2)}$ toward $\tau = -\infty$ and $\tau = +\infty$, respectively (the same is true for $-\langle p_{2b}(\tau) \rangle$ and $-\langle p_{2a}(\tau) \rangle$) while $\langle \sigma_b(\tau) \rangle$ and $\langle \sigma_a(\tau) \rangle$ are identical and vanishingly small toward $\tau = \pm\infty$. Furthermore, we observe that there is continuity between $\langle p_{1/2b}(\tau) \rangle$ and $\langle p_{1/2a}(\tau) \rangle$ and between $\langle \sigma_b(\tau) \rangle$ and $\langle \sigma_a(\tau) \rangle$ at the junction, $\tau = 0$ (see Figure 2).

2. Correlation vs. Entanglement: Connection Established via Scattering and Purity

Intuitively, if the particles are not correlated (i.e. $r = 0$) after collision, then no entanglement should be present. In this scenario, the two particle system would not experience any loss of purity so that $\mathcal{P} = 1$. Indeed, this is verified by (95). From (96) the case $\mathcal{P} = 1$ requires $a_s = 0$ or $\Sigma = 0$, that is to say, no scattering. For low energy s -wave scattering, $f(k) \approx -a_s$ and $\theta(k) = -ka_s + \mathcal{O}(k^2)$. Thus one can readily determine the requirement necessary to satisfy $a_s = 0$ or $\Sigma = 0$, namely

$$\theta = 0. \quad (107)$$

Equation (107) implies the s -wave scattering phase shift must vanish if our system is non-correlated after collision.

A question that now arises is how to determine the scattering phase shift in view of the fact that our statistical model is correlated after collision. Initially, we need to examine how correlations affect the momentum geodesic curve $\langle p_{1/2}(\tau) \rangle$. For this purpose we define the momentum-difference curve $\langle p(\tau) \rangle \equiv \frac{1}{2}[\langle p_2(\tau) \rangle - \langle p_1(\tau) \rangle]$. Comparison of the following two equations, which follow from (101), (102) and (104), (105),

$$\langle p(\tau; 0) \rangle \equiv \frac{1}{2}[\langle p_{2b}(\tau) \rangle - \langle p_{1b}(\tau) \rangle] = \sqrt{p_o^2 + 2\sigma_o^2} \tanh(A_o \tau), \quad (108)$$

$$\langle p(\tau; r) \rangle \equiv \frac{1}{2}[\langle p_{2a}(\tau) \rangle - \langle p_{1a}(\tau) \rangle] = \sqrt{(1-r)(p_o^2 + 2\sigma_o^2)} \tanh(A_o \tau), \quad (109)$$

Figure 2: Plots of $\langle p_1(\tau) \rangle$, $\langle p_2(\tau) \rangle$ and $\langle \sigma(\tau) \rangle$ before and after collision

indicates that at any arbitrary time $\tau \geq 0$

$$\langle p(\tau; 0) \rangle \geq \langle p(\tau; r) \rangle, \quad (110)$$

while both (108) and (109) share the functional argument $A_o\tau$. Condition (110) implies that the correlation causes the momentum to reduce for any $\tau \geq 0$ (relative to the non-correlated situation). This situation is analogous to the change in momentum caused by a repulsive scattering potential (see (99) and (100)). It is then reasonable to assume there exists some connection between the scattering potential and the correlation. Provided this connection is established, one should be able to determine the scattering phase shift in terms of the correlation via equations (98), (99) and (100). In this way, one can ultimately establish a connection between quantum entanglement and the statistical micro-correlation.

Recall that before collision (at the affine time $-\tau_o$) particles 1 and 2 are separated by a linear distance R_o . Each particle has momenta p_o and $-p_o$, respectively and the same momentum spread σ_o . Then from (101), (102) and (103) we have

$$p_o = \langle p_{1b}(-\tau_o) \rangle = -\langle p_{2b}(-\tau_o) \rangle = \sqrt{p_o^2 + 2\sigma_o^2} \tanh(A_o\tau_o), \quad (111)$$

$$\sigma_o = \langle \sigma_b(-\tau_o) \rangle = \sqrt{\frac{1}{2}p_o^2 + \sigma_o^2} \frac{1}{\cosh(A_o\tau_o)}. \quad (112)$$

To give an estimate of how large $A_o\tau_o$ is, we assume our momentum-space wave-packets initially have very narrow widths compared to their momenta such that $\sigma_o/p_o \sim 10^{-3}$, for example. Then by (87) we find $A_o\tau_o \sim 7.254329369$. Using (111), we find $p_o \sim 0.999999 \times \sqrt{p_o^2 + 2\sigma_o^2}$, which is equivalent to $\sigma_o/p_o \sim 10^{-3}$.

For arbitrary $\tau \geq 0$ after collision, the system of particles 1 and 2, which initially carried momenta p_o and $-p_o$, respectively at $\tau = -\tau_o$ before collision, now carries the relative conjugate-momentum $\langle p(\tau; r) \rangle$ given by (109) due to the correlation. As discussed above, through (109), (108) and (110), it is reasonable to expect the existence of a connection between the correlation and the scattering potential. With non-vanishing micro-correlation the wave-packets experience the effect of a repulsive potential; the magnitude of the wave vectors (or momenta) decreases relative to the corresponding non-correlated value. One may rewrite (98), (99) and (100) as

$$k_r \cot(k_r L) = k_o \cot(k_o L + \theta_o), \quad (113)$$

with

$$k_r = \frac{\sqrt{2\mu(\mathcal{E} - V)}}{\hbar}, \quad 0 < x < L, \quad (114)$$

$$k_o = \frac{\sqrt{2\mu\mathcal{E}}}{\hbar}, \quad x > L, \quad (115)$$

where $\theta_o \equiv \theta(k_o) \approx -k_o a_s = -p_o a_s / \hbar$ denotes the s -wave scattering phase shift, and k_r and k_o represent the wave vectors with and without the correlation, respectively. The connection between the correlation and the scattering potential can be established by combining (114) and (115).

From (110) one finds that the correlation renders

$$k_o \longrightarrow k_r \equiv \sqrt{1-r} k_o. \quad (116)$$

Then using (114), (115) and (116), we determine the scattering potential,

$$V = r\mathcal{E} = r \frac{\hbar^2 k_o^2}{2\mu} = r \frac{p_o^2}{2\mu}. \quad (117)$$

Equation (117) clearly establishes a connection between the correlation coefficient and the scattering potential: the correlation coefficient is the ratio of the scattering potential to the initial relative kinetic energy of the system. From (117) it is evident that our interaction potential is repulsive, i.e. $V > 0$ since we consider non-negative micro-correlations, $r \in [0, 1]$.

With the potential determined, one can determine the scattering phase shift by combining equations (113), (114), (115) and (117). By solving (113) for θ_o , we find

$$\tan \theta_o = \frac{k_o \tan(k_r L) - k_r \tan(k_o L)}{k_r + k_o \tan(k_o L) \tan(k_r L)}. \quad (118)$$

Substituting (116) into (118) and expanding the expression in $k_o L$ and r at the same time, one obtains

$$\tan \theta_o \approx \left[-\frac{1}{3} (k_o L)^3 + \frac{1}{15} (k_o L)^5 + \mathcal{O}[(k_o L)^7] \right] r + \left[\frac{2}{15} (k_o L)^5 + \mathcal{O}[(k_o L)^7] \right] r^2 + \mathcal{O}(r^6). \quad (119)$$

For low energy s -wave scattering, $k_o L = p_o L / \hbar \ll 1$, one may reduce (119) to

$$\tan \theta_o \approx \theta_o \approx -\frac{r (k_o L)^3}{3}. \quad (120)$$

By means of (117) and (120) we can express the scattering phase shift in terms of the scattering potential

$$\theta_o \approx -\frac{2\mu V k_o L^3}{3\hbar^2} = -\frac{2\mu V p_o L^3}{3\hbar^3}, \quad (121)$$

which is in agreement with [65].

As the scattering potential has been determined, so too can the scattering amplitude be determined. To this end, we write

$$f(k_o) = \frac{e^{i\theta_o} \sin \theta_o}{k_o} \approx \frac{\theta_o}{k_o} \approx -a_s \quad (122)$$

for low energy s -wave scattering, $k_o L = p_o L / \hbar \ll 1$. Then the squared modulus of (122), by means of (121), reads

$$|f(k_o)|^2 \approx \frac{\theta_o^2}{k_o^2} \approx \frac{r^2 k_o^4 L^6}{9} = \frac{4\mu^2 V^2 L^6}{9\hbar^4} \approx a_s^2. \quad (123)$$

Thus, we finally obtain the scattering cross section:

$$\Sigma = 4\pi |f(k_o)|^2 \approx \frac{4\pi r^2 k_o^4 L^6}{9} = \frac{16\pi \mu^2 V^2 L^6}{9\hbar^4} \approx 4\pi a_s^2. \quad (124)$$

Equations (95) and (96) above demonstrate how the entanglement can be measured from the loss of purity by use of the scattering length or cross section. By combining (95) and (123) we find the purity

$$\mathcal{P} \approx 1 - \frac{8rk_o^2 (2k_o^2 + \sigma_{k_o}^2) R_o L^3}{3} = 1 - \frac{16\mu V (2k_o^2 + \sigma_{k_o}^2) R_o L^3}{3\hbar^2}. \quad (125)$$

The correlation coefficient r can now be expressed in terms of the physical quantities such as the scattering potential, the scattering cross section and the purity. Solving equations (117), (124) and (125) for r , we obtain

$$r = \frac{V}{\mathcal{E}} = \frac{2\mu V}{\hbar^2 k_o^2} = \frac{2\mu V}{p_o^2}, \quad (126)$$

$$\approx \frac{3\sqrt{\Sigma}}{2\sqrt{\pi}k_o^2 L^3}, \quad (127)$$

$$\approx \frac{3(1 - \mathcal{P})}{8k_o^2(2k_o^2 + \sigma_{ko}^2) R_o L^3}. \quad (128)$$

In view of (41), (91), (124) and (127), one obtains the following relation:

$$\frac{V}{L^3} = \frac{4\hbar^2 k_o^4 (2k_o^2 + \sigma_{ko}^2) R_o}{3\mu}, \quad (129)$$

which indicates that the uniform scattering potential density is solely determined by the initial conditions of the given system.

From (108), (109) and (110) it is observed that for the micro-correlated Gaussian system considered here, more time is required to attain the same momentum value compared with the non-correlated Gaussian system. For example, in order to attain the same value as the initial momentum p_o , the non-correlated system and the micro-correlated system would require time intervals τ_o and τ_* , respectively, where

$$p_o = \sqrt{p_o^2 + 2\sigma_o^2 \tanh(A_o \tau_o)}, \quad (130)$$

$$p_o = \sqrt{(1 - r)(p_o^2 + 2\sigma_o^2) \tanh(A_o \tau_*)}. \quad (131)$$

Combining (130) and (131), we obtain

$$\tanh(A_o \tau_*) = (1 - r)^{-1/2} \tanh(A_o \tau_o). \quad (132)$$

Rewriting and expanding both sides of (132), we have

$$1 - 2e^{-2A_o \tau_*} + \mathcal{O}(e^{-4A_o \tau_*}) = (1 - r)^{-1/2} [1 - 2e^{-2A_o \tau_o} + \mathcal{O}(e^{-4A_o \tau_o})]. \quad (133)$$

Rounding (133) off and arranging terms,

$$e^{-2A_o(\tau_* - \tau_o)} \approx (1 - r)^{-1/2} - \frac{1}{2} [(1 - r)^{-1/2} - 1] e^{2A_o \tau_o}. \quad (134)$$

The first term on the right hand side of (134) can be approximated to 1 since $(1 - r)^{-1/2} = 1 + \frac{1}{2}r + \mathcal{O}(r^2)$ and $r \ll 1$. However, r in the second term should not be disregarded in the same way because $[(1 - r)^{-1/2} - 1] e^{2A_o \tau_o} = [\frac{1}{2}r + \mathcal{O}(r^2)] e^{2A_o \tau_o}$ is not negligible. Therefore, we may rewrite (134) as

$$e^{-2A_o \Delta} \approx 1 - [(1 - r)^{-1/2} - 1] \cdot \eta_\Delta, \quad (135)$$

where $\Delta \equiv \tau_* - \tau_o$ represents a new quantity that we term ‘‘prolongation’’, and $\eta_\Delta \equiv \frac{1}{2}e^{2A_o \tau_o} = \left(\frac{p_o}{\sigma_o}\right)^2 \exp\left[\left(\frac{\sigma_o}{p_o}\right)^2 - \frac{3}{4}\left(\frac{\sigma_o}{p_o}\right)^4 + \mathcal{O}\left[\left(\frac{\sigma_o}{p_o}\right)^6\right]\right]$ for $\frac{\sigma_o}{p_o} \ll 1$ due to (87). From (135) we find

$$\Delta \propto \left| \ln \left\{ 1 - [(1 - r)^{-1/2} - 1] \cdot \eta_\Delta \right\} \right|. \quad (136)$$

At this juncture we emphasize the following points:

- The upper bound value of r depends on the initial conditions, namely p_o and σ_o through the right-hand side of (135). The right-hand side of (135) must always be positive, so that given $r \ll 1$, we require

$$r < \frac{2}{\eta_\Delta} = 2 \left(\frac{\sigma_o}{p_o}\right)^2 \exp \left[- \left(\frac{\sigma_o}{p_o}\right)^2 + \frac{3}{4} \left(\frac{\sigma_o}{p_o}\right)^4 + \mathcal{O} \left[\left(\frac{\sigma_o}{p_o}\right)^6 \right] \right] \quad (137)$$

for $\frac{\sigma_o}{p_o} \ll 1$. For example, with $\sigma_o/p_o \sim 10^{-3}$, for the right-hand side to be positive we must have $r \lesssim 2 \times 10^{-6}$. In view of (125), equation (137) provides a lower bound estimate of the purity for a system with well-localized wave-packets, i.e. $\frac{\sigma_o}{p_o} \ll 1$.

- With r being close to the upper bound value, Δ would be infinitely large due to (136). On the other hand, with r vanishing, i.e. no correlation, Δ would vanish. This implies that Δ may serve as an indicator of quantum entanglement.
- With r held fixed, Δ depends on the initial conditions p_o and σ_o through (135).

From the above points one may infer that the prolongation Δ could represent the duration of quantum entanglement for a given micro-correlated system and that further, the duration can be controlled by the initial conditions p_o , σ_o and the micro-correlation coefficient r . From (135) it is anticipated that the maximum duration would be obtained when r is the greatest, i.e. the micro-correlation is the strongest and the ratio σ_o/p_o is the smallest (see Figure 3). We emphasize that the prolongation serves to quantify the time required by a micro-correlated system - relative to a

Figure 3: $\Delta \equiv \tau_* - \tau_o$ for different values of r : r_1 (the first plot) $<$ r_2 (the second plot)

corresponding non-correlated one - to attain the same momentum value (relative to the same initial reference time). The occurrence of a non-vanishing prolongation is in fact due to the existence of micro-correlations and therefore, due to the existence of scattering phase shifts. In other words, in the absence of scattering there is no time difference. This can be stated in yet another way as follows: “The prolongation encodes information about how long it would take an entangled system to overcome the momentum gap (relative to a corresponding non-entangled system) generated by the scattering phase shift. The entangled system only attains the full value of momentum (i.e. the momentum value as seen in the corresponding non-entangled system) when the scattering phase shift vanishes. For this reason, the prolongation represents the temporal duration over which the entanglement is active”.

From (126), (127) and (128) we observe that the micro-correlation coefficient r is directly associated with the quantum scattering process, and thus with the quantum entanglement. For example, the cross term $\langle p_1 p_2 \rangle$ in the definition of the micro-correlation coefficient r may represent the average interference between transmitted/reflected modes in the momentum degrees of freedom of particles 1 and 2. This may be viewed from a different perspective when considering the definition of micro-correlations (138). For our statistical system in which $\sigma_{p_1} = \sigma_{p_2} = \sigma$, the micro-correlation coefficient reads

$$r = r(p_1, p_2) \stackrel{\text{def}}{=} \frac{\langle p_1 p_2 \rangle - \langle p_1 \rangle \langle p_2 \rangle}{\sigma^2} \text{ with } \sigma = \sqrt{\langle (p_{1/2} - \langle p_{1/2} \rangle)^2 \rangle}. \quad (138)$$

Here the numerator is defined as covariance

$$\text{Cov}(p_1, p_2) \stackrel{\text{def}}{=} \langle p_1 p_2 \rangle - \langle p_1 \rangle \langle p_2 \rangle, \quad (139)$$

and does not vanish if the statistical system is micro-correlated. In other words, if our statistical system models a quantum scattering process, then the relevant physical information such as scattering potential and scattering cross section should be encoded in $\text{Cov}(p_1, p_2)$.

With $r \ll 1$, we may split our micro-correlated information geometry (70) into two pieces,

$$g_{ab} = \frac{1}{\sigma^2} \begin{pmatrix} 1 + r^2 + \mathcal{O}(r^3) & -r + \mathcal{O}(r^3) & 0 \\ -r + \mathcal{O}(r^3) & 1 + r^2 + \mathcal{O}(r^3) & 0 \\ 0 & 0 & 4 \end{pmatrix} = g_{\text{O}ab} + h_{ab}, \quad (140)$$

where

$$g_{\text{O}ab} = \frac{1}{\sigma^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad (141)$$

and

$$h_{ab} = \frac{1}{\sigma^2} \begin{pmatrix} r^2 + \mathcal{O}(r^3) & -r + \mathcal{O}(r^3) & 0 \\ -r + \mathcal{O}(r^3) & r^2 + \mathcal{O}(r^3) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (142)$$

This decomposition of the micro-correlated information geometry may provide a different (inherently IG) perspective on the phenomenon of quantum entanglement. In this view, the non-correlated geometry (141) is perturbed due to the presence of the quantum scattering, the information of which is encoded in (142). Thus, the quantum entanglement manifests as this information geometric perturbation of the statistical space.

V. CHAOTICITY, INFORMATION GEOMETRIC COMPLEXITY AND ENTROPY

In this Section we obtain exact expressions for the information geometric analogue of standard indicators of chaos such as sectional curvatures, Jacobi field intensities and Lyapunov exponents. Finally, we present an analytical estimate of the information geometric entropy (IGE). This will lead us to uncover connections between quantum entanglement and the complexity of informational geodesic flows in a quantitative manner.

A. Chaoticity

1. Curvatures of the Statistical Manifold $\mathcal{M}_{\text{corr.}}^{3D}$.

The Riemann curvature tensor \mathcal{R}_{abcd} of the statistical manifold $\mathcal{M}_{\text{corr.}}^{3D}$, is defined in the usual manner as [62]

$$\mathcal{R}_{bcd}^a = \partial_c \Gamma_{bd}^a - \partial_d \Gamma_{bc}^a + \Gamma_{fc}^a \Gamma_{bd}^f - \Gamma_{fd}^a \Gamma_{bc}^f, \quad (143)$$

where the non-vanishing connection coefficients are given in (77). The non-vanishing components of the Riemann tensor read

$$\mathcal{R}_{1212} = \frac{1}{4\sigma^4(r^2 - 1)}, \quad \mathcal{R}_{1313} = \frac{1}{\sigma^4(r^2 - 1)}, \quad \mathcal{R}_{1323} = -\frac{r}{\sigma^4(r^2 - 1)}, \quad \mathcal{R}_{2323} = \frac{1}{\sigma^4(r^2 - 1)}. \quad (144)$$

The Ricci curvature tensor \mathcal{R}_{ab} of the manifold $\mathcal{M}_{\text{corr.}}^{3D}$, is defined as

$$\mathcal{R}_{ab} = \partial_c \Gamma_{ab}^c - \partial_b \Gamma_{ac}^c + \Gamma_{ab}^c \Gamma_{cd}^d - \Gamma_{ac}^d \Gamma_{bd}^c. \quad (145)$$

The non-vanishing components of the Ricci tensor read

$$\mathcal{R}_{11} = \frac{1}{2\sigma^2(r^2 - 1)}, \quad \mathcal{R}_{12} = -\frac{r}{2\sigma^2(r^2 - 1)} = \mathcal{R}_{21}, \quad \mathcal{R}_{22} = \frac{1}{2\sigma^2(r^2 - 1)}, \quad \mathcal{R}_{33} = -\frac{2}{\sigma^2}. \quad (146)$$

Finally, we compute the Ricci scalar curvature \mathcal{R} of the manifold $\mathcal{M}_{\text{corr.}}^{3D}$,

$$\mathcal{R} = \mathcal{R}_a^a = \mathcal{R}_{ab} g^{ab} = -\frac{3}{2} = \sum_{i \neq j} K(e_i, e_j). \quad (147)$$

That is, the scalar curvature is the sum of all sectional curvatures $K(e_i, e_j)$ of planes spanned by pairs of orthonormal basis elements $\{e_a = \partial_{\vartheta^a(p)}\}$ of the tangent space $T_p \mathcal{M}_{\text{corr.}}^{3\text{D}}$ with $p \in \mathcal{M}_{\text{corr.}}^{3\text{D}}$. [62, 66],

$$K_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}(u, v) = \frac{\mathcal{R}_{abcd} u^a v^b u^c v^d}{(g_{ad}g_{bc} - g_{ac}g_{bd}) u^a v^b u^c v^d}; \quad u \rightarrow h^i, \quad v \rightarrow h^j \text{ with } i \neq j, \quad (148)$$

where $\langle e_a, h^b \rangle = \delta_a^b$. Notice that $K_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}$ completely determines the curvature tensor. The components of the sectional curvature are given by

$$K_{\mu_1} = -\frac{1}{4} = K_{-\mu_1}, \quad K_{\mu_2} = -\frac{1}{4} = K_{-\mu_2}, \quad K_{\sigma} = -\frac{1}{4} = K_{-\sigma}. \quad (149)$$

From above, it is worthwhile to note that both statistical manifolds $\mathcal{M}_{\text{corr.}}^{3\text{D}}$ and $\mathcal{M}_{\text{non-corr.}}^{3\text{D}}$ are negatively curved, with the micro-correlation independent Ricci scalar curvature $\mathcal{R}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}} = -\frac{3}{2} = \mathcal{R}_{\mathcal{M}_{\text{non-corr.}}^{3\text{D}}}$ and sectional curvature $K_{\mathcal{M}_{\text{corr.}}^{3\text{D}}} = -\frac{1}{4} = K_{\mathcal{M}_{\text{non-corr.}}^{3\text{D}}}$. Moreover, the constancy of the sectional curvature in all directions imply that both $\mathcal{M}_{\text{corr.}}^{3\text{D}}$ and $\mathcal{M}_{\text{non-corr.}}^{3\text{D}}$ are isotropic manifolds. Below, this will be verified by the vanishing of all components of the Weyl projective curvature tensor \mathcal{W}_{abcd} defined on each space.

2. Anisotropy and the Weyl Projective Tensor

The anisotropy of the manifold underlying system dynamics plays a crucial role in the mechanism of instability. In particular, fluctuating sectional curvatures require also that the manifold be anisotropic. The Weyl projective tensor quantifies such anisotropy and is defined as

$$\mathcal{W}_{abcd} \stackrel{\text{def}}{=} \mathcal{R}_{abcd} - \frac{1}{n-1} (\mathcal{R}_{bd}g_{ac} - \mathcal{R}_{bc}g_{ad}), \quad (150)$$

where n is the dimension of the manifold on which \mathcal{W}_{abcd} is defined. By direct computation using (70), (144) and (146), we find that all components of (150) with $n = 3$ are vanishing. The fact that $\mathcal{W}_{abcd} = 0$ implies the manifold $\mathcal{M}_{\text{corr.}}^{3\text{D}}$ is isotropic. If the manifold over which a system evolves is maximally symmetric, then

$$\mathcal{R}_{ab} = \frac{\mathcal{R}}{n} g_{ab}. \quad (151)$$

This is obtained from (150), using the fact that $\mathcal{W}_{abcd} = 0$. By inspection of (146), (70) and the fact that $\mathcal{R} = -\frac{3}{2}$ and $n = 3$, it is evident that (151) is indeed valid for our statistical manifold $\mathcal{M}_{\text{corr.}}^{3\text{D}}$. Upon substitution of (151) into (150) we obtain

$$\mathcal{W}_{abcd} = \mathcal{R}_{abcd} - \frac{\mathcal{R}}{n(n-1)} (g_{bd}g_{ac} - g_{bc}g_{ad}). \quad (152)$$

By the fact that $\mathcal{W}_{abcd} = 0$, this leads to

$$\mathcal{R}_{abcd} = \frac{\mathcal{R}}{n(n-1)} (g_{bd}g_{ac} - g_{bc}g_{ad}), \quad (153)$$

which again proves to be true for our manifold $\mathcal{M}_{\text{corr.}}^{3\text{D}}$ by inspecting (144), (70) and the fact that $\mathcal{R} = -\frac{3}{2}$ and $n = 3$. Contracting the both sides of (151), one finds

$$\delta_a^a = n, \quad (154)$$

which is also the case for our manifold $\mathcal{M}_{\text{corr.}}^{3\text{D}}$, that is, $n = 3$.

3. Jacobi Fields and Lyapunov Exponents

For the sake of clarity, consider the behavior of a family of neighboring geodesics $\left\{ \vartheta_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}^a(\tau; \vec{\varsigma}) \right\}_{\vec{\varsigma} \in \mathbb{R}_+^3}^{a=1,2,3}$ on the statistical manifold $\mathcal{M}_{\text{corr.}}^{3\text{D}}$, where τ and $\vec{\varsigma} = (\varsigma^1, \varsigma^2, \varsigma^3)$ are affine parameters. The geodesics $\vartheta_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}^a(\tau; \vec{\varsigma})$ are solutions

of equation (75). The relative geodesic spread on $\mathcal{M}_{\text{corr}}^{\text{3D}}$ is characterized by the Jacobi-Levi-Civita (JLC) equation [66, 67],

$$\frac{D^2 J^a}{D\tau^2} + \mathcal{R}_{bcd}^a \frac{\partial \vartheta^b}{\partial \tau} J^c \frac{\partial \vartheta^d}{\partial \tau} = 0, \quad (155)$$

where $a, b, c, d = 1, 2, 3$, and the second order covariant derivatives $\frac{D^2 J^a}{D\tau^2}$ are given by [68]

$$\frac{D^2 J^a}{D\tau^2} = \frac{d^2 J^a}{d\tau^2} + 2\Gamma_{bc}^a \frac{dJ^b}{d\tau} \frac{d\vartheta^c}{d\tau} + \Gamma_{bc}^a J^b \frac{d^2 \vartheta^c}{d\tau^2} + \Gamma_{bc,d}^a \frac{d\vartheta^d}{d\tau} \frac{d\vartheta^c}{d\tau} J^b + \Gamma_{bc}^a \Gamma_{df}^b \frac{d\vartheta^f}{d\tau} \frac{d\vartheta^c}{d\tau} J^d, \quad (156)$$

and the Jacobi vector field components J^a are given by

$$J^a = \delta_{\zeta} \vartheta^a \equiv \left. \frac{\partial \vartheta^a(\tau; \vec{\zeta})}{\partial \zeta^b} \right|_{\tau} \delta \zeta^b. \quad (157)$$

$\mathbf{J} = \{J^a\}_{a=1,2,3}$ represents how geodesics are separating. The JLC equation of geodesic deviation is a complicated second-order system of linear ordinary differential equations. It describes the geodesic spread on curved manifolds of a pair of nearby freely falling particles traveling on trajectories $\vartheta^a(\tau)$ and $\vartheta'^a(\tau) \stackrel{\text{def}}{=} \vartheta^a(\tau) + \delta \vartheta^a(\tau)$. Equation (155) forms a system of three coupled ordinary differential equations *linear* in the components of the deviation vector field (157) but *nonlinear* in derivatives of the metric tensor $g_{ab}(\Theta)$. It describes the linearized geodesic flow: the linearization ignores the relative velocity of the geodesics. When the geodesics are neighboring but their relative velocity is arbitrary, the corresponding geodesic deviation equation is the so-called generalized Jacobi equation [69]. The nonlinearity is due to the existence of velocity-dependent terms in the system. Neighboring geodesics accelerate relative to each other with a rate directly measured by the curvature tensor \mathcal{R}_{abcd} .

By means of (153) the second term on the light-hand side of equation (155) can be rewritten as

$$\mathcal{R}_{abcd} \frac{\partial \vartheta^b}{\partial \tau} J^c \frac{\partial \vartheta^d}{\partial \tau} = \frac{\mathcal{R} \|\mathbf{v}\|^2}{n(n-1)} P_{ab} J^b, \quad (158)$$

where $\|\mathbf{v}\| \equiv \sqrt{g_{ab} v^a v^b}$ with $v^a \equiv \frac{\partial \vartheta^a}{\partial \tau}$, and $P_{ab} \equiv g_{ab} - u_a u_b$ with $u^a \equiv v^a / \|\mathbf{v}\|$ and $u^a u_a = 1$. Due to the orthogonality between u_a and J^a , we have $P_{ab} J^b = J_a$ and thus (158) is now reduced to

$$\mathcal{R}_{abcd} \frac{\partial \vartheta^b}{\partial \tau} J^c \frac{\partial \vartheta^d}{\partial \tau} = \frac{\mathcal{R} \|\mathbf{v}\|^2}{n(n-1)} J_a. \quad (159)$$

Using equations (70) and (80), the squared modulus of v^a is computed as

$$\begin{aligned} \|\mathbf{v}\|^2 &= g_{ab} \frac{\partial \vartheta^a}{\partial \tau} \frac{\partial \vartheta^b}{\partial \tau} \\ &= \frac{1}{(1-r^2)\sigma^2} \left[\left(\frac{\partial \mu_1}{\partial \tau} \right)^2 + \left(\frac{\partial \mu_2}{\partial \tau} \right)^2 \right] - \frac{r}{(1-r^2)\sigma^2} \left(\frac{\partial \mu_1}{\partial \tau} \right) \left(\frac{\partial \mu_2}{\partial \tau} \right) + \frac{4}{\sigma^2} \left(\frac{\partial \sigma}{\partial \tau} \right)^2 \\ &= -\frac{4}{\sigma} \left[\frac{\partial^2 \sigma}{\partial \tau^2} - \frac{2}{\sigma} \left(\frac{\partial \sigma}{\partial \tau} \right)^2 \right]. \end{aligned} \quad (160)$$

Upon substitution of (86) into (160) we find

$$\|\mathbf{v}\|^2 = 4A_0^2, \quad (161)$$

where A_0 is given by (87). By combining (155), (159) and (161) we finally simplify the JLC equation to the following form:

$$\frac{D^2 J^a}{D\tau^2} + Q J^a = 0, \quad (162)$$

where

$$Q \equiv \frac{\mathcal{R} \|\mathbf{v}\|^2}{n(n-1)} = -A_0^2 < 0, \quad (163)$$

which has been computed with $n = 3$, $\mathcal{R} = -\frac{3}{2}$ and $\|\mathbf{v}\|^2$ from (161).

The Jacobi vector field intensity is given by

$$\mathcal{J}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}} \equiv \|\mathbf{J}\| = (g_{ab}J^aJ^b)^{\frac{1}{2}} = (J^aJ_a)^{\frac{1}{2}}. \quad (164)$$

For applications of the asymptotic temporal behavior of $\mathcal{J}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}(\tau)$ as a reliable indicator of chaoticity, we refer to our previous articles in references [28–30]. Defining the operator $\hat{\Omega} \equiv D^2/D\tau^2$ and observing its action on $\mathcal{J}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}^2$ leads to conclude

$$\hat{\Omega}\mathcal{J}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}^2 = \left(\hat{\Omega}J^a\right)J_a + J^a\left(\hat{\Omega}J_a\right) = -2QJ^aJ_a = -2Q\mathcal{J}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}^2. \quad (165)$$

Equation (165) follows from (162) and (164). We may however, write $\hat{\Omega}\mathcal{J}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}^2 = 2\mathcal{J}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}(\hat{\Omega}\mathcal{J}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}})$. This fact together with (165) enables the further reduction of (162) to a scalar form

$$\frac{D^2\mathcal{J}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}}{D\tau^2} + Q\mathcal{J}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}} = 0. \quad (166)$$

Since $Q < 0$, the solutions of equation (166) assume the form

$$\mathcal{J}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}(\tau) = \frac{1}{\sqrt{-Q}}\omega(0)\sinh\left(\sqrt{-Q}\tau\right), \quad (167)$$

where $\omega(0) \equiv \left.\frac{d\mathcal{J}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}(\tau)}{d\tau}\right|_{\tau=0}$.

Recalling the definition of the hyperbolic sine function, $\sinh x = \frac{1}{2}(e^x - e^{-x})$, it is clear that the geodesic deviation on $\mathcal{M}_{\text{corr.}}^{3\text{D}}$ is described by means of an exponentially divergent Jacobi vector field intensity $\mathcal{J}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}$, a classical feature of chaos. In this Riemannian geometric approach, the quantity $\lambda_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}$ defined as [47]

$$\lambda_{\mathcal{M}_{\text{corr.}}^{3\text{D}}} \stackrel{\text{def}}{=} \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \ln \left[\frac{\left|\mathcal{J}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}(\tau)\right|^2 + \left|\frac{d\mathcal{J}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}(\tau)}{d\tau}\right|^2}{\left|\mathcal{J}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}(0)\right|^2 + \left|\frac{d\mathcal{J}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}(\tau)}{d\tau}\right|_{\tau=0}^2} \right], \quad (168)$$

would play the role of the conventional Lyapunov exponents. In order to evaluate (168) we use (167) to find $\left|\mathcal{J}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}(\tau)\right|^2 = \frac{\omega^2(0)}{-Q}\sinh^2(\sqrt{-Q}\tau)$ and $\left|d\mathcal{J}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}(\tau)/d\tau\right|^2 = \omega^2(0)\cosh^2(\sqrt{-Q}\tau)$. Thus, for the case being considered,

$$\begin{aligned} \lambda_{\mathcal{M}_{\text{corr.}}^{3\text{D}}} &= \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \ln \left[-\frac{1}{Q}\sinh^2(\sqrt{-Q}\tau) + \cosh^2(\sqrt{-Q}\tau) \right] \\ &= \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \ln \left[\frac{(1-Q)}{4}e^{2\sqrt{-Q}\tau} \right] = 2\sqrt{-Q}. \end{aligned} \quad (169)$$

Therefore, it follows that

$$\lambda_{\mathcal{M}_{\text{corr.}}^{3\text{D}}} \stackrel{\tau \rightarrow \infty}{=} 2\sqrt{-Q} = 2A_o > 0, \quad (170)$$

which is due to (163). From (170) we observe the following points: the information about chaoticity encoded in the positive Lyapunov exponent does not depend on the statistical correlation, i.e. $\lambda_{\mathcal{M}_{\text{corr.}}^{3\text{D}}} = \lambda_{\mathcal{M}_{\text{non-corr.}}^{3\text{D}}} \equiv \lambda_{\mathcal{M}^{3\text{D}}} = 2A_o$, and the Lyapunov exponents can be determined solely from the initial conditions (see equation (87)).

B. Information Geometric Complexity and Entropy

We recall that a suitable indicator of temporal complexity within the IGAC framework is provided by the *information geometric entropy* (IGE) $\mathcal{S}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}(\tau)$ [45, 46],

$$\mathcal{S}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}(\tau) \stackrel{\text{def}}{=} \lim_{\tau \rightarrow \infty} \ln \mathcal{V}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}} \left[\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau) \right]. \quad (171)$$

The *information geometric complexity* (IGC) is defined as the temporal average of the dynamical statistical volume,

$$\mathcal{V}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}} \left[\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau) \right] \stackrel{\text{def}}{=} \lim_{\tau \rightarrow \infty} \left(\frac{1}{\tau} \int_0^\tau d\tau' \text{vol} \left[\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau') \right] \right). \quad (172)$$

The extended volume $\text{vol} \left[\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau') \right]$ of the effective parameter space explored by the system at time τ' is given by

$$\text{vol} \left[\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau') \right] \stackrel{\text{def}}{=} \int \int \int_{\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau')} \rho_{(\mathcal{M}_{\text{corr.}}^{3\text{D}}, g)}(\vartheta^1, \vartheta^2, \vartheta^3) d\vartheta^1 d\vartheta^2 d\vartheta^3, \quad (173)$$

where $\rho_{(\mathcal{M}_{\text{corr.}}^{3\text{D}}, g)}(\vartheta^1, \vartheta^2, \vartheta^3)$ is the so-called Fisher density and equals the square root of the determinant $g = |\det(g_{ab})|$ of the metric tensor $g(\vartheta^1, \vartheta^2, \vartheta^3)$,

$$\rho_{(\mathcal{M}_{\text{corr.}}^{3\text{D}}, g)}(\vartheta^1, \vartheta^2, \vartheta^3) \stackrel{\text{def}}{=} \sqrt{g(\vartheta^1, \vartheta^2, \vartheta^3)}. \quad (174)$$

The set $\mathcal{D}_{\Theta}^{(\text{geodesic})}$ represents a subspace of the whole (permitted) parameter space $\mathcal{D}_{\Theta}^{(\text{total})}$ in (50),

$$\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau') = \{ \Theta \equiv (\vartheta^1, \vartheta^2, \vartheta^3) : \vartheta^a(0) \leq \vartheta^a \leq \vartheta^a(\tau') \}, \quad (175)$$

where $a = 1, 2, 3$, and $\vartheta^a \equiv \vartheta^a(s)$ with $0 \leq s \leq \tau'$ such that $\vartheta^a(s)$ satisfies (75). The elements of $\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau')$ are the macrovariables $\{\Theta\}$ whose components ϑ^a are bounded by specified limits of integration $\vartheta^a(0)$ and $\vartheta^a(\tau')$. The limits of integration are obtained via integration of the set of coupled nonlinear second order ordinary differential equations characterizing the geodesic equations. In the case of the statistical manifold of three-dimensional Gaussian probability distributions parametrized in terms of $\Theta = (\mu_1, \mu_2, \sigma)$, the integration space $\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau')$ in (173) is the direct product of the parameter subspaces \mathcal{I}_{μ_1} , \mathcal{I}_{μ_2} and \mathcal{I}_{σ} , where in the Gaussian case, $\mathcal{I}_{\mu_1} = (-\infty, +\infty)_{\mu_1}$, $\mathcal{I}_{\mu_2} = (-\infty, +\infty)_{\mu_2}$ and $\mathcal{I}_{\sigma} = (0, +\infty)_{\sigma}$ such that

$$\mathcal{D}_{\Theta}^{(\text{geodesic})} = \mathcal{I}_{\mu_1} \otimes \mathcal{I}_{\mu_2} \otimes \mathcal{I}_{\sigma} = [(-\infty, +\infty) \otimes (-\infty, +\infty) \otimes (0, +\infty)]. \quad (176)$$

In the IGAC, we are interested in a probabilistic description of the evolution of a given system in terms of its corresponding probability distribution on $\mathcal{M}_{\text{corr.}}^{3\text{D}}$, which is homeomorphic to $\mathcal{D}_{\Theta}^{(\text{geodesic})}$. We are interested in the evolution of the system from $\tau_{\text{initial}} = 0$ to $\tau_{\text{final}} = \tau$. Within the probabilistic description, investigating the evolution of the system from $\tau_{\text{initial}} = 0$ to $\tau_{\text{final}} = \tau$ is equivalent to studying the shortest path (or, in terms of the ME method [52, 53, 57, 58], the maximally probable path) leading from $\Theta(0)$ to $\Theta(\tau)$.

Formally, the IGE $\mathcal{S}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}(\tau)$ is defined in terms of an averaged parametric 3-fold integral (τ is the parameter) over the three-dimensional geodesic paths connecting $\Theta(0)$ to $\Theta(\tau)$. In the present IG approach, the IGC represents a statistical measure of complexity of the macroscopic path $\Theta \stackrel{\text{def}}{=} \Theta(\tau)$ on $\mathcal{M}_{\text{corr.}}^{3\text{D}}$, connecting initial and final macrostates Θ_{I} and Θ_{F} , respectively. The path $\Theta(\tau)$ is obtained via integration of the geodesic equation on $\mathcal{M}_{\text{corr.}}^{3\text{D}}$, generated by the universal ME updating method. At a discrete level, the path $\Theta(\tau)$ can be described in terms of an infinite continuous sequence of intermediate macroscopic states, $\Theta(\tau) = [\Theta_{\text{I}}, \dots, \Theta_{\bar{k}-1}, \Theta_{\bar{k}}, \Theta_{\bar{k}+1}, \dots, \Theta_{\text{F}}]$ with $\Theta_{\bar{j}} = \Theta(\tau_{\bar{j}})$, determined via the logarithmic relative entropy maximization procedure subjected to appropriately-specified normalization and information constraints. The nature of such constraints defines the (correlational) structure of the underlying probability distribution on the particular curved statistical manifold $\mathcal{M}_{\text{corr.}}^{3\text{D}}$. In other words, the correlational structure that emerges in our IG statistical models originates in the information pertaining to the microscopic degrees of freedom of the actual physical systems. It is finally quantified in terms of the intuitive notion of volume growth via the IGC or alternatively in entropic terms by the IGE. The IGC is then interpreted as the temporally averaged volume of the statistical macrospace explored by the system, in the asymptotic limit, in its evolution from Θ_{I} to Θ_{F} . Otherwise, upon a suitable normalization procedure that makes the IGC an adimensional quantity, it represents the number of accessible macrostates (with coordinates living in the accessible parameter space $\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau)$) explored by the system in its evolution from Θ_{I} to Θ_{F} .

The temporal average in (172) has been introduced in order to smear out the possibly very complex fine details of the entropic dynamical description of the system on $\mathcal{M}_{\text{corr.}}^{3\text{D}}$. Thus, we provide a coarse-grained-like inferential description of the system's chaotic dynamics. The long-term asymptotic temporal behavior is adopted in order to properly characterize dynamical indicators of chaoticity (for instance, Lyapunov exponents, entropies, etc.) eliminating transient effects which enters the computation of the expected value of (173). In chaotic transients, one observes that

typical initial conditions behave in an apparently chaotic manner for a possibly long time, but then asymptotically approach a non-chaotic attractor in a rapid fashion.

In the case under investigation, (172) is given by

$$\mathcal{V}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}} \left[\mathcal{D}_{\Theta}^{(\text{geodesic})} (\tau; r) \right] = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^{\tau} d\tau' \int_{\mu_1(0)}^{\mu_1(\tau')} \int_{\mu_2(0)}^{\mu_2(\tau')} \int_{\sigma(0)}^{\sigma(\tau')} \sqrt{g} d\mu_1 d\mu_2 d\sigma, \quad (177)$$

where the geodesic paths $\Theta(\tau) = (\mu_1(\tau; r), \mu_2(\tau; r), \sigma(\tau; r))$ are given in (84), (85) and (86) and the determinant of the information metric reads

$$g = \det(g_{ab}) = \frac{4}{(1-r^2)\sigma^6}. \quad (178)$$

Substituting (178) into (177), and evaluating the integral by means of (84), (85) and (86), we obtain the IGC for the correlated Gaussian statistical models:

$$\begin{aligned} \mathcal{V}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}} \left[\mathcal{D}_{\Theta}^{(\text{geodesic})} (\tau; r) \right] &= -\frac{1}{\sqrt{1-r^2}} \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^{\tau} d\tau' \left[\mu_1(\tau'') \Big|_{\tau''=0}^{\tau''=\tau'} \mu_2(\tau'') \Big|_{\tau''=0}^{\tau''=\tau'} \frac{1}{\sigma^2(\tau'')} \Big|_{\tau''=0}^{\tau''=\tau'} \right] \\ &= \frac{8\sqrt{\frac{1-r}{1+r}}}{\lambda_{\mathcal{M}^{3\text{D}}}} \left[-\frac{3}{4} \lambda_{\mathcal{M}^{3\text{D}}} + \frac{1}{4} \frac{\sinh(\lambda_{\mathcal{M}^{3\text{D}}}\tau)}{\tau} + \frac{\tanh(\frac{1}{2}\lambda_{\mathcal{M}^{3\text{D}}}\tau)}{\tau} \right], \end{aligned} \quad (179)$$

where A_0 has been replaced with $\frac{1}{2}\lambda_{\mathcal{M}^{3\text{D}}}$ due to (170). For non-correlated Gaussian statistical models the IGC becomes

$$\mathcal{V}_{\mathcal{M}_{\text{non-corr.}}^{3\text{D}}} \left[\mathcal{D}_{\Theta}^{(\text{geodesic})} (\tau; 0) \right] = \frac{8}{\lambda_{\mathcal{M}^{3\text{D}}}} \left[-\frac{3}{4} \lambda_{\mathcal{M}^{3\text{D}}} + \frac{1}{4} \frac{\sinh(\lambda_{\mathcal{M}^{3\text{D}}}\tau)}{\tau} + \frac{\tanh(\frac{1}{2}\lambda_{\mathcal{M}^{3\text{D}}}\tau)}{\tau} \right]. \quad (180)$$

Inserting (179) into (171) and working through some calculations, we obtain the IGE for the correlated Gaussian statistical models:

$$\mathcal{S}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}} (\tau; r) \stackrel{\tau \rightarrow \infty}{\rightarrow} \lambda_{\mathcal{M}^{3\text{D}}}\tau - \ln(\lambda_{\mathcal{M}^{3\text{D}}}\tau) + \frac{1}{2} \ln\left(\frac{1-r}{1+r}\right). \quad (181)$$

For non-correlated Gaussian statistical models the IGE becomes

$$\mathcal{S}_{\mathcal{M}_{\text{non-corr.}}^{3\text{D}}} (\tau; 0) \stackrel{\tau \rightarrow \infty}{\rightarrow} \lambda_{\mathcal{M}^{3\text{D}}}\tau - \ln(\lambda_{\mathcal{M}^{3\text{D}}}\tau). \quad (182)$$

By means of (179) and (180) we compare the asymptotic (long-time limit) expressions of the IGCs in the presence and absence of micro-correlations, respectively, to obtain

$$\frac{\mathcal{V}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}} \left[\mathcal{D}_{\Theta}^{(\text{geodesic})} (\tau; r) \right]}{\mathcal{V}_{\mathcal{M}_{\text{non-corr.}}^{3\text{D}}} \left[\mathcal{D}_{\Theta}^{(\text{geodesic})} (\tau; 0) \right]} = \sqrt{\frac{1-r}{1+r}}. \quad (183)$$

From (181) and (182) we also find

$$\mathcal{S}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}} (\tau; r) - \mathcal{S}_{\mathcal{M}_{\text{non-corr.}}^{3\text{D}}} (\tau; 0) = \frac{1}{2} \ln\left(\frac{1-r}{1+r}\right). \quad (184)$$

From (183) and (184) we find that both the IGC and the IGE decrease in presence of micro-correlations. In particular, the IGC decreases by the factor $\sqrt{\frac{1-r}{1+r}} < 1$ for $r > 0$ whereas the IGE decreases by $\frac{1}{2} \ln\left(\frac{1-r}{1+r}\right) < 0$ for $r > 0$.

It is evident from (184) that in presence of micro-correlations the IGE is attenuated in a correlation-dependent manner: $\mathcal{S}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}} (\tau; r)$ decreases as the magnitude of the correlation increases. It is important to observe that this has no relation to the asymptotic (long-time limit) feature of the IGE. The correlated IGE is reduced by $\frac{1}{2} \ln\left(\frac{1-r}{1+r}\right) < 0$

for $r > 0$, which is independent of the evolution of the system (see equation (184)). When the micro-correlations vanish (i.e. $r = 0$), we obtain the expected result $\mathcal{S}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}(\tau; r) = \mathcal{S}_{\mathcal{M}_{\text{non-corr.}}^{3\text{D}}}(\tau; 0)$.

With $\mathcal{V}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}}[\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau; r)]$ in hand, we make the following observations. From (183) we find

$$r = \frac{\Delta\mathcal{C}^2}{\mathcal{C}_{\text{total}}^2}, \quad (185)$$

where

$$\Delta\mathcal{C}^2 \equiv \left\{ \mathcal{V}_{\mathcal{M}_{\text{non-corr.}}^{3\text{D}}} \left[\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau; 0) \right] \right\}^2 - \left\{ \mathcal{V}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}} \left[\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau; r) \right] \right\}^2 \quad (186)$$

and

$$\mathcal{C}_{\text{total}}^2 \equiv \left\{ \mathcal{V}_{\mathcal{M}_{\text{non-corr.}}^{3\text{D}}} \left[\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau; 0) \right] \right\}^2 + \left\{ \mathcal{V}_{\mathcal{M}_{\text{corr.}}^{3\text{D}}} \left[\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau; r) \right] \right\}^2. \quad (187)$$

Combining (125) and (185), we obtain

$$\mathcal{P} = 1 - \eta_{\mathcal{C}} \cdot \frac{\Delta\mathcal{C}^2}{\mathcal{C}_{\text{total}}^2}, \quad (188)$$

where the dimensionless coefficient $\eta_{\mathcal{C}} \equiv \frac{8}{3}k_o^2(2k_o^2 + \sigma_{k_o}^2)R_oL^3$. From (188) it is evident that quantum entanglement and the information geometric complexity are connected. It turns out that when purity goes to unity, the difference between the correlated and non-correlated information geometric complexities approaches zero.

VI. FINAL REMARKS

In this article, micro-correlated and non-correlated Gaussian statistical models were used to model the entanglement of a quantum mechanical system generated by an s -wave scattering event. The IGAC was used to analyze our specific two-variable micro-correlated Gaussian statistical model. The manifolds $\mathcal{M}_{\text{corr.}}^{3\text{D}}$ and $\mathcal{M}_{\text{non-corr.}}^{3\text{D}}$ were used to model the quantum entanglement induced by head-on elastic scattering of two spinless, structureless, non-relativistic particles, each represented by minimum uncertainty wave-packets. The degree of entanglement was quantified by the purity of the system. The purity \mathcal{P} for s -wave scattering was found in terms of the micro-correlation coefficient r , the interaction potential range L , the initial separation R_o between particles, the initial momentum $p_o = \hbar k_o$ and initial momentum spread $\sigma_o = \hbar \sigma_{k_o}$. The scattering phase shift θ as well as the scattering cross section Σ were both found to be defined in terms of r , L and p_o . For $r = 0$, θ , Σ and V (interaction potential height) are each zero while $\mathcal{P} = 1$ (indicating that the system is not entangled). The micro-correlation coefficient r , a quantity that parameterizes the correlated microscopic degrees of freedom of the system, can be understood as the ratio of the potential to kinetic energy of the system. When $r \neq 0$ the wave-packets experience the effect of a repulsive potential; the magnitude of the wave vectors (momenta) decreases relative to their corresponding non-correlated value. The upper bound value of r depends on p_o and σ_o in such a manner that r increases as p_o decreases. This result constitutes a significant, explicit connection between micro-correlations (the correlation coefficient r) and physical observables (the macrovariable p_o). The role played by r in the quantities \mathcal{P} , Σ , θ , and V suggests that information about quantum scattering and therefore about quantum entanglement is encoded in the statistical micro-correlation, specifically in the covariance term $\text{Cov}(p_1, p_2) \stackrel{\text{def}}{=} \langle p_1 p_2 \rangle - \langle p_1 \rangle \langle p_2 \rangle$ appearing in the definition (138) of r .

In summary, we proposed that the emergence of scattering-induced quantum entanglement can be understood by considering pre and post-collisional quantum dynamical scenarios as macroscopic manifestations emerging from appropriately chosen statistical microstructures. In this view, the information geometry associated with the post-collisional statistical microstructure can be modelled in terms of a weak perturbation of the information geometry relative to the pre-collisional microstructure. In particular, quantum entanglement may be interpreted as a perturbation of statistical space geometry: the non-correlated geometry (141) is perturbed due to the presence of the quantum scattering, the information of which is encoded in the statistical micro-correlation terms present in (142). Indeed, in the case where $r = 0$, the perturbation matrix (142) is null. Thus, the quantum entanglement manifests as a geometric perturbation of the statistical space in analogy to the interpretation of a static gravitational field as a perturbation of flat space. The perturbation of statistical geometry occurs in the 2D momentum subspace spanned by basis vectors $e_1 = \partial_{\mu_1}$ and $e_2 = \partial_{\mu_2}$. In particular, after scattering the two particles maintain a correlation among their microscopic momentum degrees of freedom regardless of the extent of their separation in statistical space. This fact, together with the time-independence of the statistical geometry [i.e. the information metric is Riemannian (rather than pseudo-Riemannian)]

since its signature is positive definite (rather than positive semi-definite)] leads to a notion of statistical non-locality. The perturbation of statistical geometry is associated with the scattering phase shift in the statistical momentum space.

The prolongation, denoted Δ , was defined as the time required for the observed momentum difference between a correlated and corresponding non-correlated system to vanish. The prolongation encodes information about how long it would take an entangled system to overcome the momentum gap generated by the scattering phase shift. The entangled system only attains the full value of momentum (i.e. the momentum value as seen in the corresponding non-correlated system) when the scattering phase shift vanishes. For this reason, the prolongation represents the temporal duration over which the entanglement is active. It was found that for r values close to its upper bound, the prolongation Δ becomes infinitely large. On the other hand, with r vanishing (i.e., no micro-correlation) Δ is identically zero. With r fixed however, the prolongation Δ depends on p_o and σ_o . Thus, the prolongation Δ may be taken to represent the duration of quantum entanglement for a given correlated system where the entanglement duration can be controlled by the initial conditions p_o and σ_o as well as r . Maximal prolongation occurs when r is greatest and the ratio σ_o/p_o is smallest. For small initial r and p_o , Δ would be correspondingly small, suggesting that for such scenarios quantum entanglement is transient.

It was determined that both statistical manifolds $\mathcal{M}_{\text{corr.}}^{3D}$ and $\mathcal{M}_{\text{non-corr.}}^{3D}$ are negatively curved, with a micro-correlation independent Ricci scalar curvature $\mathcal{R}_{\mathcal{M}_{\text{corr.}}^{3D}} = -\frac{3}{2} = \mathcal{R}_{\mathcal{M}_{\text{non-corr.}}^{3D}}$. Moreover, the sectional curvature throughout both manifolds was determined to be constant, $K_{\mathcal{M}_{\text{corr.}}^{3D}} = -\frac{1}{4} = K_{\mathcal{M}_{\text{non-corr.}}^{3D}}$. The constancy of the sectional curvature in all directions imply that both $\mathcal{M}_{\text{corr.}}^{3D}$ and $\mathcal{M}_{\text{non-corr.}}^{3D}$ are isotropic manifolds. This was verified by the vanishing of all components of the Weyl projective curvature tensor \mathcal{W}_{abcd} defined on each space. The complexity of geodesic paths on $\mathcal{M}_{\text{corr.}}^{3D}$ and $\mathcal{M}_{\text{non-corr.}}^{3D}$ was characterized through the asymptotic computation of the IGE and the Lyapunov exponents on each manifold. The Lyapunov exponents in both cases were found to be constant and positive definite, i.e. $\lambda_{\mathcal{M}_{\text{corr.}}^{3D}} = \lambda_{\mathcal{M}_{\text{non-corr.}}^{3D}} = 2A_o > 0$. The IGE $\mathcal{S}_{\mathcal{M}_{\text{corr.}}^{3D}}(\tau; r)$ in presence of micro-correlations assumes smaller values relative to the non-correlated case $\mathcal{S}_{\mathcal{M}_{\text{non-corr.}}^{3D}}(\tau; 0)$ while the growth characteristics of both correlated and non-correlated IGEs were found to be the same. Specifically, the larger the micro-correlation (i.e. the closer r is to 1) the lower the values of the IGE $\mathcal{S}_{\mathcal{M}_{\text{corr.}}^{3D}}(\tau; r)$. Thus, the stronger the micro-correlation, the larger the gap between $\mathcal{S}_{\mathcal{M}_{\text{corr.}}^{3D}}(\tau; r)$ and $\mathcal{S}_{\mathcal{M}_{\text{non-corr.}}^{3D}}(\tau; 0)$. This implies that $\mathcal{S}_{\mathcal{M}_{\text{corr.}}^{3D}}(\tau; r) < \mathcal{S}_{\mathcal{M}_{\text{non-corr.}}^{3D}}(\tau; 0)$. When micro-correlations vanish (i.e. when $r = 0$), we obtain the expected result, $\mathcal{S}_{\mathcal{M}_{\text{corr.}}^{3D}}(\tau; 0) = \mathcal{S}_{\mathcal{M}_{\text{non-corr.}}^{3D}}(\tau; 0)$. In the model investigated in this work, the appearance of micro-correlation terms in the elements in the Fisher-Rao information metric leads to the compression of $\mathcal{V}_{\mathcal{M}_{\text{corr.}}^{3D}}[\mathcal{D}_{\Theta}^{(\text{geodesic})}(\tau; r)]$ by the fraction $\sqrt{\frac{1-r}{1+r}}$ and thus, to a reduction of the complexity of the path leading from initial macrostate Θ_I to final macrostate Θ_F .

Information Geometry and Maximum Relative Entropy methods hold great promise for solving computational problems in classical and quantum physics. Our theoretical formalism allows for the analysis of physical problems by means of statistical inference and information geometric techniques, that is, Riemannian (differential) geometric techniques applied to probability theory. The macroscopic behavior of an arbitrary complex system is a consequence of the underlying statistical structure of the microscopic degrees of freedom of the system. We are confident that the present work represents significant progress toward the goal of understanding the relationship between statistical micro-correlations and quantum entanglement on the one hand and the effect of micro-correlations on the dynamical complexity of informational geodesic flows on the other. It is our hope to build upon the techniques employed in this work to ultimately establish a sound information geometric interpretation of quantum entanglement.

Acknowledgments

This work was partially supported by WCU (World Class University) program of NRF/MEST (R32-2009-000-10130-0) and by the European Community's Seventh Framework Program FP7/2007-2013 under grant agreement 213681 (CORNER Project).

Appendix A: Integration of the Geodesic Equations

The coupled ODEs (78), (79) and (80) can be solved via the following strategy. First, (78) and (79) can be rewritten as

$$\frac{\mu_{1/2}''}{\mu_{1/2}'} = \frac{2\sigma'}{\sigma}, \quad (\text{A1})$$

where “'” denotes a differentiation with respect to τ . (A1) can be recasted as

$$\frac{x'}{x} = \frac{2\sigma'}{\sigma}, \text{ where } x = \mu'_{1/2} \text{ and } x' = \mu''_{1/2}. \quad (\text{A2})$$

Moreover, since

$$\frac{x'}{x} = \frac{2\sigma'}{\sigma} \Rightarrow \frac{d}{d\tau} \ln |x| = 2 \frac{d}{d\tau} \ln \sigma, \quad (\text{A3})$$

we find

$$\int \left(\frac{d}{d\tau} \ln |x| \right) d\tau = 2 \int \left(\frac{d}{d\tau} \ln \sigma \right) d\tau \Rightarrow \ln |x| + k = 2 \ln \sigma, \text{ where } k = \text{const}. \quad (\text{A4})$$

Exponentiating both sides of the above equation leads to

$$\exp(\ln |x| + k) = \exp(2 \ln \sigma) \Rightarrow e^k |x| = \sigma^2. \quad (\text{A5})$$

Thus,

$$|\mu'_{1/2}| = |C_{1/2}| \sigma^2, \quad (\text{A6})$$

where $C_{1/2}$ are the integration constants corresponding to $\mu'_{1/2}$. In order for our Gaussian statistical model to have smooth and natural evolution, $\sigma(\tau)$ must be positive definite and well-behaved (continuous and differentiable) over the entire domain of τ ; $\tau \in (-\infty, +\infty)$. Then from (A6) $\mu'_{1/2}$ must be either positive definite or negative definite over the entire domain of τ and the sign of $C_{1/2}$ must be associated with the sign of $\mu'_{1/2}$ so that σ is positive definite and free from nodes. We can rewrite (A6) as

$$\mu'_{1/2} = C_{1/2} \sigma^2; \text{ with } \frac{\mu'_{1/2}}{C_{1/2}} > 0. \quad (\text{A7})$$

Substituting (A6) into (80), we obtain

$$\sigma'' - \frac{\sigma'^2}{\sigma} + \frac{1}{4(r^2 - 1)} (2rC_1C_2 - C_1^2 - C_2^2) \sigma^3 = 0. \quad (\text{A8})$$

Dividing both sides of (A8) by σ yields

$$\frac{\sigma''\sigma - \sigma'^2}{\sigma^2} + \frac{1}{4(r^2 - 1)} (2rC_1C_2 - C_1^2 - C_2^2) \sigma^2 = 0. \quad (\text{A9})$$

The first term of (A9) can be rewritten as a complete differential by means of the following identity:

$$\frac{d}{d\tau} \left(\frac{\sigma'}{\sigma} \right) = \frac{\sigma''\sigma - \sigma'^2}{\sigma^2}. \quad (\text{A10})$$

The second term of (A9) is also a complete differential form due to (A6). Then, for μ_1 and μ_2 , respectively, we may rewrite (A9) as

$$\frac{d}{d\tau} \left(\frac{\sigma'}{\sigma} \right) + \frac{C_1}{4(r^2 - 1)} \left[\frac{C_2}{C_1} \left(2r - \frac{C_2}{C_1} \right) - 1 \right] \mu'_1 = 0, \quad (\text{A11})$$

$$\frac{d}{d\tau} \left(\frac{\sigma'}{\sigma} \right) + \frac{C_2}{4(r^2 - 1)} \left[\frac{C_1}{C_2} \left(2r - \frac{C_1}{C_2} \right) - 1 \right] \mu'_2 = 0. \quad (\text{A12})$$

Integrating both sides with respect to τ , these become

$$\frac{\sigma'}{\sigma} + \frac{C_1}{4(r^2 - 1)} \left[\frac{C_2}{C_1} \left(2r - \frac{C_2}{C_1} \right) - 1 \right] \mu_1 + D_1 = 0, \quad (\text{A13})$$

$$\frac{\sigma'}{\sigma} + \frac{C_2}{4(r^2 - 1)} \left[\frac{C_1}{C_2} \left(2r - \frac{C_1}{C_2} \right) - 1 \right] \mu_2 + D_2 = 0, \quad (\text{A14})$$

where D_1 and D_2 are integration constants.

Substituting (A13) and (A14) into (A1) leads to

$$\mu_1'' + \frac{C_1}{2(r^2 - 1)} \left[\frac{C_2}{C_1} \left(2r - \frac{C_2}{C_1} \right) - 1 \right] \mu_1 \mu_1' + 2D_1 \mu_1' = 0, \quad (\text{A15})$$

$$\mu_2'' + \frac{C_2}{2(r^2 - 1)} \left[\frac{C_1}{C_2} \left(2r - \frac{C_1}{C_2} \right) - 1 \right] \mu_2 \mu_2' + 2D_2 \mu_2' = 0. \quad (\text{A16})$$

Then integration of both sides of (A15) and (A16) with respect to τ yields

$$\mu_1' + \frac{C_1}{4(r^2 - 1)} \left[\frac{C_2}{C_1} \left(2r - \frac{C_2}{C_1} \right) - 1 \right] \mu_1^2 + 2D_1 \mu_1 + E_1 = 0, \quad (\text{A17})$$

$$\mu_2' + \frac{C_2}{4(r^2 - 1)} \left[\frac{C_1}{C_2} \left(2r - \frac{C_1}{C_2} \right) - 1 \right] \mu_2^2 + 2D_2 \mu_2 + E_2 = 0. \quad (\text{A18})$$

Equations (A17) and (A18) can now be represented by the general form:

$$\mu' + a\mu^2 + b\mu + c = 0, \quad (\text{A19})$$

which is known as the ‘‘Riccati equation’’ [43]. Due to the fact that a , b and c are all constants in our problem, (A19) may be modified to a more tractable form:

$$\nu' + A\nu^2 + B = 0, \quad (\text{A20})$$

where

$$\nu = \mu + \frac{b}{2a}, \quad (\text{A21})$$

$$A = a, \quad (\text{A22})$$

$$B = -\frac{b^2}{4a} + c. \quad (\text{A23})$$

The solution of (A20) is given by the form:

$$\nu = \frac{1}{A} \frac{\alpha u' + \beta v'}{\alpha u + \beta v}, \quad (\text{A24})$$

with α and β being arbitrary constants, not both zero, while u and v are linearly independent solutions of

$$\frac{d}{d\tau} \left(\frac{1}{A} \frac{dz}{d\tau} \right) + Bz = 0 \Rightarrow \frac{d^2 z}{d\tau^2} + ABz = 0. \quad (\text{A25})$$

One finds easily

$$u = e^{\gamma\tau}, \quad (\text{A26})$$

$$v = e^{-\gamma\tau}, \quad (\text{A27})$$

where

$$\gamma = \pm \sqrt{-AB} = \pm \frac{\sqrt{b^2 - 4ac}}{2}. \quad (\text{A28})$$

Then by means of equations (A21), (A22), (A23), (A24), (A26), (A27) and (A28), we find

$$\begin{aligned} \mu(\tau) &= \frac{\gamma}{a} \frac{\alpha e^{\gamma\tau} - \beta e^{-\gamma\tau}}{\alpha e^{\gamma\tau} + \beta e^{-\gamma\tau}} - \frac{b}{2a} \\ &= \pm \frac{\sqrt{b^2 - 4ac}}{2a} \frac{\alpha \exp \left[\pm \frac{\sqrt{b^2 - 4ac}}{2} \tau \right] - \beta \exp \left[\mp \frac{\sqrt{b^2 - 4ac}}{2} \tau \right]}{\alpha \exp \left[\pm \frac{\sqrt{b^2 - 4ac}}{2} \tau \right] + \beta \exp \left[\mp \frac{\sqrt{b^2 - 4ac}}{2} \tau \right]} - \frac{b}{2a}. \end{aligned} \quad (\text{A29})$$

Finally, we may identify equations (A17) and (A18) with (A19) to find the solutions μ_1 and μ_2 via (A29):

$$\mu_{1/2}(\tau) = \frac{\gamma_{1/2} \alpha_{1/2} e^{\gamma_{1/2}\tau} - \beta_{1/2} e^{-\gamma_{1/2}\tau}}{a_{1/2} \alpha_{1/2} e^{\gamma_{1/2}\tau} + \beta_{1/2} e^{-\gamma_{1/2}\tau}} - \frac{b_{1/2}}{2a_{1/2}}, \quad (\text{A30})$$

where for μ_1

$$\gamma_1 \equiv \pm \frac{\sqrt{b_1^2 - 4a_1c_1}}{2}, \quad (\text{A31})$$

with

$$a_1 = \frac{C_1}{4(r^2 - 1)} \left[\frac{C_2}{C_1} \left(2r - \frac{C_2}{C_1} \right) - 1 \right], \quad (\text{A32})$$

$$b_1 = 2D_1, \quad (\text{A33})$$

$$c_1 = E_1, \quad (\text{A34})$$

and for μ_2

$$\gamma_2 \equiv \pm \frac{\sqrt{b_2^2 - 4a_2c_2}}{2}, \quad (\text{A35})$$

with

$$a_2 = \frac{C_2}{4(r^2 - 1)} \left[\frac{C_1}{C_2} \left(2r - \frac{C_1}{C_2} \right) - 1 \right], \quad (\text{A36})$$

$$b_2 = 2D_2, \quad (\text{A37})$$

$$c_2 = E_2. \quad (\text{A38})$$

In order for our system to have non-oscillatory and non-constant evolution, $\gamma_{1/2}$ must be real, thus the quantities $b_{1/2}^2 - 4a_{1/2}c_{1/2}$ must be positive definite. Later, we will find the conditions for this (see (A53)).

We may rewrite (A30) as

$$\mu_{1/2}(\tau) = \frac{\gamma_{1/2} \delta_{1/2} e^{\gamma_{1/2}\tau} - e^{-\gamma_{1/2}\tau}}{a_{1/2} \delta_{1/2} e^{\gamma_{1/2}\tau} + e^{-\gamma_{1/2}\tau}} - \frac{b_{1/2}}{2a_{1/2}}, \quad (\text{A39})$$

where

$$\delta_{1/2} \equiv \frac{\alpha_{1/2}}{\beta_{1/2}}. \quad (\text{A40})$$

By means of (A7) and (A39) we find

$$\sigma(\tau) = \sqrt{\frac{\mu'_{1/2}(\tau)}{C_{1/2}}} = 2 \sqrt{\left| \frac{\delta_{1/2}}{a_{1/2}C_{1/2}} \right| \left| \frac{\gamma_{1/2}}{\delta_{1/2}e^{\gamma_{1/2}\tau} + e^{-\gamma_{1/2}\tau}} \right|}, \quad (\text{A41})$$

where $a_{1/2}$, $\gamma_{1/2}$, $\delta_{1/2}$ are given by (A32), (A36), (A31), (A35) and (A40). However, our σ obtained either via μ_1 or via μ_2 must be identical. This yields the following equality:

$$\frac{\mu'_1}{C_1} = \frac{\mu'_2}{C_2} \Leftrightarrow \frac{4\delta_1\gamma_1^2}{a_1C_1(\delta_1e^{\gamma_1\tau} + e^{-\gamma_1\tau})^2} = \frac{4\delta_2\gamma_2^2}{a_2C_2(\delta_2e^{\gamma_2\tau} + e^{-\gamma_2\tau})^2}. \quad (\text{A42})$$

In order for (A42) to be generally true, the following conditions must be satisfied:

$$|\gamma_1| = |\gamma_2|, \quad (\text{A43})$$

$$\delta_1 = \delta_2 = 1, \quad (\text{A44})$$

$$a_1C_1 = a_2C_2. \quad (\text{A45})$$

From (A32) and (A36) one finds that (A45) holds true by itself. In order for (A43) to hold true, we require

$$D_1^2 - \frac{C_1 E_1}{4(r^2 - 1)} \left[\frac{C_2}{C_1} \left(2r - \frac{C_2}{C_1} \right) - 1 \right] = D_2^2 - \frac{C_2 E_2}{4(r^2 - 1)} \left[\frac{C_1}{C_2} \left(2r - \frac{C_1}{C_2} \right) - 1 \right]. \quad (\text{A46})$$

Substituting the conditions (A43) and (A44) into (A39), we obtain

$$\mu_{1/2} = \frac{\gamma}{a_{1/2}} \tanh(\gamma\tau) - \frac{D_{1/2}}{a_{1/2}}, \quad (\text{A47})$$

where $\gamma = |\gamma_1| = |\gamma_2|$. Adding μ_1 and μ_2 , we find

$$\mu_1 + \mu_2 = \left(\frac{1}{a_1} + \frac{1}{a_2} \right) \gamma \tanh(\gamma\tau) - \left(\frac{D_1}{a_1} + \frac{D_2}{a_2} \right). \quad (\text{A48})$$

We make use of this Gaussian system to model a head-on collision between two Gaussian packets in momentum space, where each particle carries the average momentum, $\langle p_1 \rangle = \mu_1$ and $\langle p_2 \rangle = \mu_2$, respectively. Thus, the total momentum of the two-particle system represented by (A48) must be conserved. This requires

$$\frac{1}{a_1} + \frac{1}{a_2} = 0. \quad (\text{A49})$$

For convenience we require both $\mu_1(\tau)$ and $\mu_2(\tau)$ cross 0 at $\tau = 0$. From (A47) we find that this condition implies

$$D_1 = D_2 = 0. \quad (\text{A50})$$

From (A49) one finds

$$C_1 = -C_2 \equiv C \neq 0. \quad (\text{A51})$$

Then due to (A45), (A50) and (A51), (A46) is reduced to

$$E_1 = -E_2 \equiv E \neq 0. \quad (\text{A52})$$

Substituting (A50), (A51) and (A52) into (A46), we obtain the above mentioned reality condition for $\gamma_{1/2}$, namely

$$CE < 0. \quad (\text{A53})$$

From (A31) together with (A32), (A33), (A34), (A43), (A50), (A51), (A52), we find

$$\gamma = |\gamma_1| = |\gamma_2| = \sqrt{\frac{CE}{2(r-1)}}. \quad (\text{A54})$$

Then substituting (A50), (A51) and (A54) into (A47), and using (A41), we finally obtain

$$\mu_1(\tau; r) = -\sqrt{\frac{2E(r-1)}{C}} \tanh\left(\sqrt{\frac{CE}{2(r-1)}}\tau\right), \quad (\text{A55})$$

$$\mu_2(\tau; r) = \sqrt{\frac{2E(r-1)}{C}} \tanh\left(\sqrt{\frac{CE}{2(r-1)}}\tau\right), \quad (\text{A56})$$

$$\sigma(\tau; r) = \sqrt{-\frac{E}{C}} \frac{1}{\cosh\left(\sqrt{\frac{CE}{2(r-1)}}\tau\right)}, \quad (\text{A57})$$

where we have set $C < 0$ and $E > 0$. In our probabilistic macroscopic approach to dynamics, these geodesic trajectories represent the maximum probability paths on $\mathcal{M}_{\text{corr.}}^{3D}$.

For the non-correlated Gaussian system, we set $r = 0$ in (A55), (A56) and (A57) to obtain

$$\mu_1(\tau; 0) = -\sqrt{-\frac{2E}{C}} \tanh\left(\sqrt{-\frac{CE}{2}}\tau\right), \quad (\text{A58})$$

$$\mu_2(\tau; 0) = \sqrt{-\frac{2E}{C}} \tanh\left(\sqrt{-\frac{CE}{2}}\tau\right), \quad (\text{A59})$$

$$\sigma(\tau; 0) = \sqrt{-\frac{E}{C}} \frac{1}{\cosh\left(\sqrt{-\frac{CE}{2}}\tau\right)}. \quad (\text{A60})$$

Distinguishing the constants C and E for the correlated Gaussian system from those for the non-correlated Gaussian system, we rewrite (A55), (A56) and (A57) as

$$\mu_1(\tau; r) = -\sqrt{\frac{2E_r(r-1)}{C_r}} \tanh\left(\sqrt{\frac{C_r E_r}{2(r-1)}} \tau\right), \quad (\text{A61})$$

$$\mu_2(\tau; r) = \sqrt{\frac{2E_r(r-1)}{C_r}} \tanh\left(\sqrt{\frac{C_r E_r}{2(r-1)}} \tau\right), \quad (\text{A62})$$

$$\sigma(\tau; r) = \sqrt{-\frac{E_r}{C_r}} \frac{1}{\cosh\left(\sqrt{\frac{C_r E_r}{2(r-1)}} \tau\right)}, \quad (\text{A63})$$

where the subscript “ r ” in C_r and E_r implies that the constants are dependent upon the correlation coefficient r of the given statistical manifold.

Appendix B: Refining the Geodesic Trajectories

In this Appendix we join two different charts of Gaussian statistical manifolds, one without correlation (before collision) and the other with correlation (after collision). The set of geodesic curves for each model is represented by equations (A58), (A59), (A60) (for the non-correlated model) and by equations (A61), (A62), (A63) (for the correlated model). The two sets are joined at the junction, $\tau = 0$: $\tau < 0$ (before collision) for the non-correlated model and $\tau \geq 0$ (after collision) for the correlated model.

The constants, C and E in (A58), (A59) and (A60) can be determined via the conditions at the initial affine time, $-\tau_o$. We assign the initial momenta and the dispersion of the wave-packets as

$$\mu_1(-\tau_o; 0) = -\sqrt{-\frac{2E}{C}} \tanh\left(-\sqrt{-\frac{CE}{2}} \tau_o\right) \equiv p_o, \quad (\text{B1})$$

$$\mu_2(-\tau_o; 0) = \sqrt{-\frac{2E}{C}} \tanh\left(-\sqrt{-\frac{CE}{2}} \tau_o\right) \equiv -p_o, \quad (\text{B2})$$

$$\sigma(-\tau_o; 0) = \sqrt{-\frac{E}{C}} \frac{1}{\cosh\left(-\sqrt{-\frac{CE}{2}} \tau_o\right)} \equiv \sigma_o. \quad (\text{B3})$$

Combining (B1) and (B3), one obtains

$$-\frac{E}{C} = \frac{1}{2} p_o^2 + \sigma_o^2. \quad (\text{B4})$$

Also, taking the ratio between σ_o and p_o via (B1) and (B3) yields,

$$\frac{\sigma_o}{p_o} = \frac{1}{\sqrt{2} \sinh\left(\sqrt{-\frac{CE}{2}} \tau_o\right)}. \quad (\text{B5})$$

Upon considering large τ_o in (B5), we find

$$\begin{aligned} \sqrt{-\frac{CE}{2}} \tau_o &= \frac{1}{\tau_o} \sinh^{-1}\left(\frac{p_o}{\sqrt{2}\sigma_o}\right) \\ &\stackrel{\frac{\sigma_o}{p_o} \ll 1}{=} \frac{1}{\tau_o} \left\{ \ln\left(\frac{\sqrt{2}p_o}{\sigma_o}\right) + \frac{1}{2} \left(\frac{\sigma_o}{p_o}\right)^2 - \frac{3}{8} \left(\frac{\sigma_o}{p_o}\right)^4 + \mathcal{O}\left[\left(\frac{\sigma_o}{p_o}\right)^6\right] \right\}. \end{aligned} \quad (\text{B6})$$

Equation (B6) implies that τ_o should be chosen sufficiently large so that the ratio σ_o/p_o will be very small, while $\sqrt{-\frac{CE}{2}}$ remains finite. The constants C and E can be individually determined by simultaneously solving (B4) and (B6).

In a similar manner, the constants C_r and E_r in (A61), (A62) and (A63) can be determined via the conditions at the reversal time τ_o . We assign the momenta and dispersion of the wave-packets according to

$$\mu_1(\tau_o; r) = -\sqrt{\frac{2E_r(r-1)}{C_r}} \tanh\left(\sqrt{\frac{C_r E_r}{2(r-1)}} \tau_o\right) \equiv -p'_o, \quad (\text{B7})$$

$$\mu_2(\tau_o; r) = \sqrt{\frac{2E_r(r-1)}{C_r}} \tanh\left(\sqrt{\frac{C_r E_r}{2(r-1)}} \tau_o\right) \equiv p'_o, \quad (\text{B8})$$

$$\sigma(\tau_o; r) = \sqrt{-\frac{E_r}{C_r}} \frac{1}{\cosh\left(\sqrt{\frac{C_r E_r}{2(r-1)}} \tau_o\right)} \equiv \sigma'_o. \quad (\text{B9})$$

Combination of (B7) with (B9) leads to

$$-\frac{E_r}{C_r} = \frac{p_o'^2}{2(1-r)} + \sigma_o'^2. \quad (\text{B10})$$

From (B7) and (B9) it is found that the ratio between σ'_o and p'_o reads

$$\frac{\sigma'_o}{p'_o} = \frac{1}{\sqrt{2(1-r)}} \frac{1}{\sinh\left(\sqrt{\frac{C_r E_r}{2(r-1)}} \tau_o\right)}. \quad (\text{B11})$$

From (B11) it is found that for large τ_o ,

$$\begin{aligned} \sqrt{\frac{C_r E_r}{2(r-1)}} &= \frac{1}{\tau_o} \sinh^{-1}\left(\frac{p'_o}{\sqrt{2(1-r)}\sigma'_o}\right) \\ &\stackrel{\frac{\sigma'_o}{p'_o} \ll 1}{=} \frac{1}{\tau_o} \left\{ \ln\left(\frac{\sqrt{2}p'_o}{\sqrt{1-r}\sigma'_o}\right) + \frac{1-r}{2} \left(\frac{\sigma'_o}{p'_o}\right)^2 - \frac{3}{8} (1-r)^2 \left(\frac{\sigma'_o}{p'_o}\right)^4 + \mathcal{O}\left[\left(\frac{\sigma'_o}{p'_o}\right)^6\right] \right\}. \end{aligned} \quad (\text{B12})$$

Here again it is implied that τ_o should be taken sufficiently large so that the ratio, σ'_o/p'_o can be very small while $\sqrt{\frac{C_r E_r}{2(r-1)}}$ remains finite. Furthermore, the constants C_r and E_r can be individually determined by simultaneously solving (B10) and (B12).

The two sets of geodesic curves (with and without correlations) are joined at the junction $\tau = 0$. The two sets of geodesic curves must be continuous at the junction $\tau = 0$ so as to ensure the collision does not assume any unphysical irregularity in the momentum dispersion. From (A58), (A59), (A60) and (A61), (A62), (A63) it is found that this continuity condition is satisfied by

$$\frac{E}{C} = \frac{E_r}{C_r}. \quad (\text{B13})$$

Using condition (B13) together with (B1) and (B7), one may compare p_o with p'_o as follows,

$$\begin{aligned} \frac{p'_o}{p_o} &= \sqrt{1-r} \frac{\tanh\left(\sqrt{\frac{C_r E_r}{2(r-1)}} \tau_o\right)}{\tanh\left(\sqrt{-\frac{CE}{2}} \tau_o\right)} \\ &= \sqrt{1-r} \left\{ 1 + 2(\epsilon - \epsilon') + \mathcal{O}\left[(\epsilon - \epsilon')^2\right] \right\}, \end{aligned} \quad (\text{B14})$$

where $\epsilon \equiv \exp(-\sqrt{-2CE}\tau_o)$ and $\epsilon' \equiv \exp\left(-\sqrt{\frac{2C_r E_r}{r-1}} \tau_o\right)$. In a similar manner, by way of (B3) and (B9) one may also compare σ_o with σ'_o ,

$$\begin{aligned} \frac{\sigma'_o}{\sigma_o} &= \frac{\cosh\left(\sqrt{-\frac{CE}{2}} \tau_o\right)}{\cosh\left(\sqrt{\frac{C_r E_r}{2(r-1)}} \tau_o\right)} \\ &= \sqrt{\frac{\epsilon'}{\epsilon}} \left\{ 1 + (\epsilon - \epsilon') + \mathcal{O}\left[(\epsilon - \epsilon')^2\right] \right\}. \end{aligned} \quad (\text{B15})$$

From (B14) it is observed that for sufficiently large τ_o the ratio p'_o/p_o is not significantly influenced by how the functional arguments $\sqrt{-\frac{CE}{2}}\tau_o$ and $\sqrt{\frac{C_r E_r}{2(r-1)}}\tau_o$ compare with each other, since the quantities on the right-hand side of ϵ and ϵ' are very small (as is the difference $\epsilon - \epsilon'$). From (B15) however, the ratio σ'_o/σ_o appears to be influenced by how those functional arguments compare with each other since the leading approximation reads

$$\sqrt{\frac{\epsilon'}{\epsilon}} = \exp \left[\left(\sqrt{-\frac{CE}{2}} - \sqrt{\frac{C_r E_r}{2(r-1)}} \right) \tau_o \right]. \quad (\text{B16})$$

From (B16), one observes that the difference $\sqrt{-\frac{CE}{2}} - \sqrt{\frac{C_r E_r}{2(r-1)}}$ must vanish in order for $\sqrt{\epsilon'/\epsilon}$ to remain finite given that τ_o is sufficiently large; otherwise a non-vanishing difference could result in a sufficiently large exponent when multiplied by a large value of τ_o - this would cause $\sqrt{\epsilon'/\epsilon}$ to grow or decay exponentially. A vanishing value of $\sqrt{-\frac{CE}{2}} - \sqrt{\frac{C_r E_r}{2(r-1)}}$ implies

$$\epsilon = \epsilon'. \quad (\text{B17})$$

From (B14), (B15) and (B17) it follows that

$$p'_o = \sqrt{1-r} p_o, \quad (\text{B18})$$

$$\sigma'_o = \sigma_o. \quad (\text{B19})$$

Equations (B18) and (B19) also satisfies the condition (B13) through (B4) and (B10).

Substituting (B4), (B6) and (B10), (B12) into (A58), (A59), (A60) and (A61), (A62), (A63), respectively, and using (B17), (B18), (B19), we may rewrite the geodesic trajectories as follows: for the non-correlated Gaussian system,

$$\mu_1(\tau; 0) = -\sqrt{p_o^2 + 2\sigma_o^2} \tanh(A_o \tau), \quad (\text{B20})$$

$$\mu_2(\tau; 0) = \sqrt{p_o^2 + 2\sigma_o^2} \tanh(A_o \tau), \quad (\text{B21})$$

$$\sigma(\tau; 0) = \sqrt{\frac{1}{2}p_o^2 + \sigma_o^2} \frac{1}{\cosh(A_o \tau)}, \quad (\text{B22})$$

while for the correlated Gaussian system,

$$\mu_1(\tau; r) = -\sqrt{(1-r)(p_o^2 + 2\sigma_o^2)} \tanh(A_o \tau), \quad (\text{B23})$$

$$\mu_2(\tau; r) = \sqrt{(1-r)(p_o^2 + 2\sigma_o^2)} \tanh(A_o \tau), \quad (\text{B24})$$

$$\sigma(\tau; r) = \sqrt{\frac{1}{2}p_o^2 + \sigma_o^2} \frac{1}{\cosh(A_o \tau)}, \quad (\text{B25})$$

where

$$A_o \equiv \sqrt{-\frac{CE}{2}} = \frac{1}{\tau_o} \sinh^{-1} \left(\frac{p_o}{\sqrt{2}\sigma_o} \right). \quad (\text{B26})$$

$$\stackrel{\frac{\sigma_o}{p_o} \ll 1}{=} \frac{1}{\tau_o} \left\{ \ln \left(\frac{\sqrt{2}p_o}{\sigma_o} \right) + \frac{1}{2} \left(\frac{\sigma_o}{p_o} \right)^2 - \frac{3}{8} \left(\frac{\sigma_o}{p_o} \right)^4 + \mathcal{O} \left[\left(\frac{\sigma_o}{p_o} \right)^6 \right] \right\},$$

which is defined from (B6).

-
- [1] A. Einstein, B. Podolsky, N. Rosen, Phys. Rev. **47**, 777 (1935).
 - [2] E. Schrödinger, Naturwiss. **48**, 807 1935; **49**, 823 1935; **50**, 844 1935.
 - [3] C. K. Law, Phys. Rev. **A70**, 062311 (2004).
 - [4] O. Kübler and H. D. Zeh, Ann. Phys. (N.Y.) **76**, 405 (1973).
 - [5] W. Dur, G. Vidal, J. I. Cirac, N. Linden, and S. Popescu, Phys. Rev. Lett. **87**, 137901 (2001).
 - [6] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Information*, Cambridge University Press (2000).

- [7] S. L. Braunstein and P. van Loock, *Rev. Mod. Phys.* **77**, 513 (2005).
- [8] A. Furusawa, J. L. Sørensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble and E. S. Polzik, *Science* **282**, 706 (1998).
- [9] Ch. Silberhorn, P. K. Lam, O. Weiss, F. König, N. Korolkova, and G. Leuchs, *Phys. Rev. Lett.* **86**, 4267 (2001).
- [10] Z. Y. Ou, S. F. Pereira, H. J. Kimble, and K. C. Peng, *Phys. Rev. Lett.* **68**, 3663 (1992).
- [11] K. W. Chan, J. P. Torres, and J. H. Eberly, *Phys. Rev.* **A75**, 050101 R (2007).
- [12] R. Grobe, K. Rzazewski, and J. H. Eberly, *J. Phys.* **B27**, L503 (1994).
- [13] W.-C. Liu, J. H. Eberly, S. L. Haan, and R. Grobe, *Phys. Rev. Lett.* **83**, 520 (1999).
- [14] N. Chandra and R. Ghosh, *Phys. Rev.* **A69**, 012315 (2004).
- [15] M. V. Fedorov, M. A. Efremov, A. E. Kazakov, K. W. Chan, C. K. Law, and J. H. Eberly, *Phys. Rev.* **A69**, 052117 (2004).
- [16] G. Morigi, J. Eschner, S. Mancini and D. Vitali, *Phys. Rev. Lett.* **96**, 023601 (2006).
- [17] P. A. Miller and S. Sarkar, *Phys. Rev.* **E60**, 1542 (1999); A. Tanaka, H. Fujisaki, and T. Miyadera, *ibid.* **66**, 045201 (2002); H. Fujisaki, T. Miyadera, and A. Tanaka, *ibid.* **67**, 066201 (2003); X.-W. Hou and B. Hu, *Phys. Rev.* **A69**, 042110 (2004).
- [18] Ph. Jacquod, *Phys. Rev. Lett.* **92**, 150403 (2004).
- [19] M. Busschardt and M. Freyberger, *Phys. Rev.* **A75**, 052101 (2007).
- [20] A. Tal and G. Kurizki, *Phys. Rev. Lett.* **94**, 160503 (2005).
- [21] L. S. Schulman, *Phys. Rev.* **A57**, 840 (1998).
- [22] H. Mack and M. Freyberger, *Phys. Rev.* **A66**, 042113 (2002).
- [23] L. S. Schulman, *Phys. Rev. Lett.* **92**, 210404 (2004).
- [24] J. Wang, C. K. Law, and M.-C. Chu, *Phys. Rev.* **A73**, 034302 (2006).
- [25] J. Wang, C. K. Law, and M.-C. Chu, *Phys. Rev.* **A72**, 022346 (2005).
- [26] F. Schmuser and D. Janzing, *Phys. Rev.* **A73**, 052313 (2006).
- [27] G. Benenti and G. Casati, *Phys. Rev.* **E79**, 025201 (2009).
- [28] C. Cafaro and S. A. Ali, *EJTP* **5**, 139 (2008).
- [29] C. Cafaro and S. A. Ali, *Physica* **D234**, 70 (2007).
- [30] C. Cafaro and S. A. Ali, *Physica* **A387**, 6876 (2008).
- [31] S. A. Ali, C. Cafaro, D.-H. Kim, S. Mancini, *Physica* **A389**, 3117 (2010).
- [32] C. Cafaro and S. Mancini, *Phys. Scr.* **82**, 035007 (2010).
- [33] C. Cafaro, A. Giffin, S. A. Ali, D.-H. Kim, *Appl. Math. Comput.* **217**, 2944 (2010).
- [34] N. L. Harshman and G. Hutton, arXiv: quant-ph/0710.5776, (2007).
- [35] Z.-B. Chen, J.-W. Pan, G. Hou, and Y.-D. Zhang, *Phys. Rev. Lett.* **88**, 040406 (2002).
- [36] R. Simon, *Phys. Rev. Lett.* **84**, 2726 (2000).
- [37] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, *Rev. Mod. Phys.* **74**, 145 (2002).
- [38] A. Ferraro, S. Olivares, and M. G. A. Paris, arXiv: quant-ph/0503237 (2005).
- [39] W. H. Zurek, S. Habib, and J. P. Paz, *Phys. Rev. Lett.* **70**, 1187 (1993).
- [40] M. Tegmark, *Foundations of Phys. Lett.* **6**, 571 (1993).
- [41] M. Tegmark and H. S. Shapiro, *Phys. Rev.* **E50**, 2538 (1994).
- [42] D. Giulini, E. Joos, C. Kiefer, J. Kupsch, I.-O. Stamatescu, and H. D. Zeh, *Decoherence and the Appearance of a Classical World in Quantum Theory* (Springer-Verlag, Berlin, 1996).
- [43] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, San Diego (1996).
- [44] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-relativistic Theory*, Butterworth-Heinemann (1981).
- [45] C. Cafaro, “*The Information Geometry of Chaos*”, Ph. D. Thesis, SUNY at Albany, NY-USA (2008).
- [46] C. Cafaro, *Chaos, Solitons & Fractals* **41**, 886 (2009).
- [47] L. Casetti, C. Clementi, and M. Pettini, *Phys. Rev.* **E54**, 5969 (1996).
- [48] M. Di Bari and P. Cipriani, *Planet. Space Sci.* **46**, 1543 (1998).
- [49] R. F. Fox and J. Keizer, *Phys. Rev.* **A43**, 1709 (1991).
- [50] C. G. J. Jacobi, *Vorlesungen über Dynamik*, Reimer, Berlin (1866).
- [51] A. Caticha, “*Entropic Dynamics*”, in *Bayesian Inference and Maximum Entropy Methods in Science and Engineering*, ed. by R.L. Fry, AIP Conf. Proc. **617**, 302 (2002).
- [52] A. Caticha and R. Preuss, *Phys. Rev.* **E70**, 046127 (2004).
- [53] A. Giffin, “*Maximum Entropy: The Universal Method for Inference*”, Ph. D. Thesis, SUNY at Albany, NY-USA (2008).
- [54] S. Amari and H. Nagaoka, *Methods of Information Geometry*, American Mathematical Society, Oxford University Press, 2000.
- [55] A. Caticha, “*Change, Time and Information Geometry*”, in “Maximum Entropy and Bayesian Methods in Science and Engineering” ed. by A. Mohammad-Djafari, AIP Conf. Proc. **568**, 72 (2001).
- [56] C. Cafaro and S. Mancini, *Physica* **D240**, 607 (2011).
- [57] A. Caticha and A. Giffin, “*Updating Probabilities*”, in *Bayesian Inference and Maximum Entropy Methods in Science and Engineering*, ed. by Ali Mohammad-Djafari, AIP Conf. Proc. **872**, 31 (2006).
- [58] A. Caticha, “*Relative Entropy and Inductive Inference*”, *Bayesian Inference and Maximum Entropy Methods in Science and Engineering*, ed. by G. Erickson and Y. Zhai, AIP Conf. Proc. **707**, 75 (2004).
- [59] E. T. Jaynes, *Probability Theory: The Logic of Science*, Cambridge University Press (2003).
- [60] C. Cafaro, *Int. J. Theor. Phys.* **47**, 2924 (2008).
- [61] Y. A. Rozanov, *Probability Theory: A Concise Course*, Dover Publications, New York (1977).
- [62] F. De Felice and J. S. Clarke, *Relativity on Curved Manifolds*, Cambridge University Press (1990).
- [63] J. Gemmer, A. Otte, and G. Mahler, *Phys. Rev. Lett.* **86**, 1927 (2001).

- [64] The online lecture notes on Nuclear Physics II by Dr. M. R. C. Hunt, available at www.physics.gla.ac.uk.
- [65] K. Mishima, M. Hayashi and S. H. Lin, Phys. Lett. **A333**, 371 (2004).
- [66] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, Freeman & Co., San Francisco (1973).
- [67] M. P. do Carmo, *Riemannian Geometry*, Birkhauser, Boston (1992).
- [68] H. C. Ohanian and R. Ruffini, *Gravitation and Spacetime*, W.W. Norton & Company (1994).
- [69] C. Chicone and B. Mashhoon, Class. Quantum Grav. **19**, 4231 (2002).

This figure "Fig1.png" is available in "png" format from:

<http://arxiv.org/ps/1104.1250v1>

This figure "Fig2.png" is available in "png" format from:

<http://arxiv.org/ps/1104.1250v1>

This figure "Fig3.png" is available in "png" format from:

<http://arxiv.org/ps/1104.1250v1>